Constructing the tree of shapes of an image by fusion of the trees of connected components of upper and lower level sets

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Devoted to the memory of Professor H.H. Schaefer

Abstract

The tree of shapes of an image is an ordered structure which permits an efficient manipulation of the level sets of an image, modeled as a real continuous function defined on a rectangle of $\mathbb{R}^N$, $N \geq 2$. In this paper we construct the tree of shapes of an image by fusing both trees of connected components of upper and lower level sets. We analyze the branch structure of both trees and we construct the tree of shapes by joining their branches in a suitable way. This was the algorithmic approach for 2D images introduced by F. Guichard and P. Monasse in their initial paper, though other efficient approaches were later developed in this case. In this paper, we prove the well-foundedness of this approach for the general case of multidimensional images. This approach can be effectively implemented in the case of 3D images and can be applied for segmentation, but this is not the object of this paper.

Key words: tree of shapes, order complete, level sets

1 Introduction

Ordered tree structures play an important role in several contexts in image processing. They permit a hierarchical organization of information and the development of fast algorithms. Let us review some of them.

In most image processing based applications, an image is usually viewed as a set of pixels placed on a rectangular grid. The pixel provides a very local information: taking it as elementary unit places the scale of representation far from the interpretation or decision scale. In recent years, an increasing number of applications rely on more structured image representations. For instance, region-based or level-lines image representations offer two advantages with respect to pixel based ones: the number of regions or level-lines is much lower than the number of original pixels, and they represent a first level of abstraction with respect to the raw image information.

Let us mention two data structures which have proved to be useful in region based image processing: the region adjacency graph and tree based structures. The region adjacency graph (RAG) is the right data structure needed to encode a partition of the image domain: the nodes represent regions and two nodes are connected by an edge if their associated regions are neighbors. This structure is well adapted to many segmentation algorithms: starting with an

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initial partition organized as a RAG, and merging regions according to an homogeneity criterion we derive another partition represented by a RAG. Since this structure can only represent a single scale of the image, other tree based structures have been developed to represent a hierarchy of partitions. Quadtrees, or Partition Trees [26] are examples of such structures.

One of the most sound alternatives to pixel based representations of images comes from mathematical morphology. Mathematical morphology can be considered as the study of the lattice of real functions defined in a given domain [29] and the operators acting on them. On the other hand, it addresses the problem of the basic atoms of information that describe the image. According to Mathematical Morphology, an image $u : \Omega \to I\!R$, $\Omega \subseteq I\!R^N$, is representative of an equivalence class of images v obtained from u via a contrast change, i.e., $v = g(u)$ where g, for simplicity, will be a continuous strictly increasing function [13, 29] modeling global illumination changes. The contrast of an image depends on the sensor's properties, on the lightning conditions, on the objects' reflection properties, etc., and these conditions are usually unknown. This led the physicist and gestaltist M. Wertheimer [34] to state as a principle that the grey level is not an observable: images are observed up to an arbitrary and unknown contrast change. Mathematical Morphology recognized contrast invariance as a basic requirement and proposed that image analysis operations should take into account this invariance principle [29]. Under this assumption, an image is characterized by its level sets (see Section 3) which constitute the basic objects (atoms) for image processing and analysis. Later, in order to account for local changes in illumination, several authors [6, 27, 29] proposed a more local description of the basic objects of an image, more precisely, they proposed to consider the connected components of (upper or lower) level sets as basic objects of the image.

In many cases, for 2D images, a connected component of a level set can be described in terms of its boundaries which, by an abuse of language, we call level lines. By Sard's theorem, this is the case if the image u is a smooth function, but more general cases can be included with the right definition of level lines, i.e., as boundaries of level sets. The family of level lines of the image can be given an ordered tree structure since they are ordered by inclusion. This is essentially the tree of shapes of the image which has been implemented for 2D images in [24, 22] (and [32] using an slightly different approach). It gives a complete and non-redundant representation of the image and is contrast independent. The tree of shapes merges into a single tree the information contained in the trees of connected components of upper ($\{x : u(x) \geq \lambda\}$, $\lambda \in I\!R$) and lower ($\{x : u(x) \leq \lambda\}$) level sets.

In the case of 2D images, many image processing tasks like edge detection, segmentation, or registration have been restricted to the family of level lines (or its equivalent formulation by the tree of shapes). Indeed, edge detection computed as a subfamily of level lines has been the object of several works [12], and the computed edges have been used for recognition purposes [20, 5]. Level lines have also been used efficiently for segmentation [3], or registration [23, 20]. All these works deal with two-dimensional images. In order to extend this approach to the multidimensional case, we study in this paper the ordered structure of the family of connected components of level sets of a continuous function defined in a domain of $I\!R^N$ $(N \geq 2)$ homeomorphic to the unit ball and justify mathematically the algorithmic approach to the construction of the tree of shapes of the image. This order data structure encodes in an efficient way the family of its level surfaces as the external boundaries of its upper and lower level sets. Motivated by its applications to segmentation, it has been implemented for three-dimensional images in [21]. Figures 1 and 2 show a Section of a 3D CT angiography image and some of its

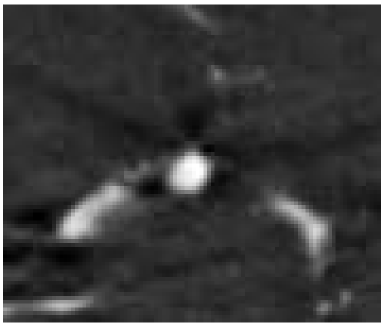

Figure 1: A Section of a 3D CT angiography image from which we display some level surfaces in Figure 2.

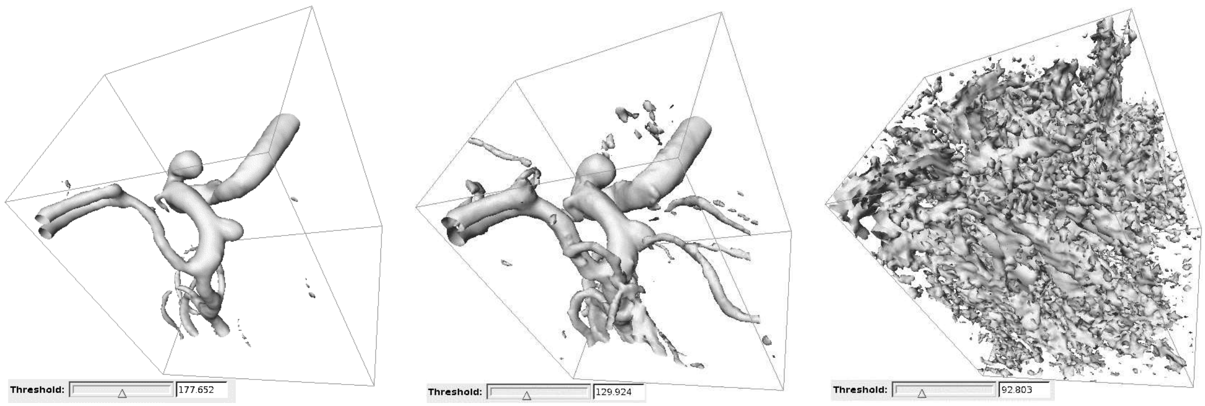

Figure 2: Three level surfaces of the 3D image whose 2D section is displayed in Figure 1 (we used AMIRA visualization software). The tree of shapes is a data structure which is able to handle all these level surfaces and use them for several tasks in image processing like segmentation or registration.

level surfaces. The data structure for 3D images is able to handle efficiently all of them with a memory cost essentially proportional to the number of voxels of the image.

Let us finally summarize the structure of this paper. In Section 2, we introduce the basic order (trees, intervals, branches) and topological preliminaries used in the sequel. In Section 3 we introduce the trees of connected components of upper an lower level sets and the tree of shapes of an image. In Section 4 we study the order completion of the above mentioned trees, which is necessary to give (in Section 5) a detailed description of its maximal branches, the right tool to construct the tree of shapes by fusion of the trees of connected components of upper an lower level sets. In Section 6, the branch structure is interpreted in connection with the singular facts of the corresponding upper and lower tree. The construction of the tree of shapes by fusion of both upper and lower trees is the object of Section 7. Finally, Section 8 is devoted to a quick overview of the literature on data structures for images and surfaces.

2 Preliminaries

2.1 Trees as ordered structures

For basic concepts on ordered structures we refer to [28].

Definition 2.1. *Let $(\mathcal{T}, \leq)$ be an ordered set. We say that $(\mathcal{T}, \leq)$ is a tree if*

(i) $\mathcal{T}$ has a largest element,

(ii) Given $C, D \in \mathcal{T}$, the supremum, denoted by $C \vee D$ exits in $\mathcal{T}$,

(iii) If $C, D \in \mathcal{T}$ have a minorant in $\mathcal{T}$, then either $C \leq D$ or $D \leq C$. In that case we shall say that C and D are nested.

If $C, D \in \mathcal{T}$ have no minorant we shall write $C \wedge D = \emptyset$. If we add $\emptyset$ as an element of $\mathcal{T}$, then the notation $C \wedge D$ has a sense and is always equal to C, D or $\emptyset$. The elements of the tree will be called nodes. Observe that, thanks to (*iii*), there are no cycles in a tree.

Let $(\mathcal{T}, \leq)$ be a tree. As usual, if $A \leq B \in \mathcal{T}$, we define the interval $[A, B]_{\mathcal{T}}$ by

$$[A, B]_{\mathcal{T}} = \{S \in \mathcal{T} : A \leq S \leq B\}.$$

Definition 2.2. *Let $(\mathcal{T}, \leq)$ be a tree, $A, B \in \mathcal{T}$. We say that B contains a bifurcation if there are $S, T \in \mathcal{T}$, $S, T \leq B$ such that $S \wedge T = \emptyset$. We say that there is a bifurcation between A and B in $\mathcal{T}$ if there is $S \in \mathcal{T}$ such that $S \leq B$ and $S \wedge T = \emptyset$.*

Definition 2.3. *Let $(\mathcal{T}, \leq)$ be a tree, $A, B \in \mathcal{T}$. We say that $[A, B]_{\mathcal{T}}$ is a branch of $\mathcal{T}$ if there is no bifurcation between A and B.*

For simplicity, having fixed the tree $\mathcal{T}$, $[A, B]$ will denote an interval of $\mathcal{T}$.

Proposition 2.4. *Let $(\mathcal{T}, \leq)$ be a tree. Let $[A_1, B_1]$, $[A_2, B_2]$ be two branches of $\mathcal{T}$ such that*

$$[A_1, B_1] \cap [A_2, B_2] \neq \emptyset.$$

Then $[A_1 \wedge A_2, B_1 \vee B_2]$ is a branch.

Proof. Let $S \in \mathcal{T}$ be such that $A_1 \leq S \leq B_1$ and $A_2 \leq S \leq B_2$. Thus $B_1 \wedge B_2 \neq \emptyset$, and, therefore, B_1 and B_2 are nested. We observe that A_1 and A_2 are also nested. Indeed, since $A_2 \leq S \leq B_1$, and since there is no bifurcation in $[A_1, B_1]$, we have that $A_1 \wedge A_2 \neq \emptyset$. Then A_1 and A_2 are nested. By symmetry, we may assume that either

$$A_1 \leq B_1 \leq A_2 \leq B_2, \text{ or} \tag{i}$$

$$A_1 \leq A_2 \leq B_1 \leq B_2, \text{ or} \tag{ii}$$

$$A_1 \leq A_2 \leq B_2 \leq B_1. \tag{iii}$$

If (*i*) holds, then

$$A_1 \leq S \leq B_1 \leq A_2 \leq S \leq B_2,$$

and $B_1 = A_2 = S$. This case can be subsumed under the case (*ii*). If (*iii*) holds, then $[A_1 \wedge A_2, B_1 \vee B_2] = [A_1, B_1]$ is a branch. Thus, we may assume that (*ii*) holds. We prove that there is no bifurcation in $[A_1, B_2]$.

Let R a node of $\mathcal{T}$ such that $R \le B_2$. Since $[A_2, B_2]$ is a branch, $R \wedge A_2 \neq \emptyset$. By definition of tree, either $A_2 \le R$ or $R \le A_2$. In the first case, $A_1 \le R$, while, in the second case, $R \le B_1$ and, since $[A_1, B_1]$ is a branch, $R \wedge A_1 \neq \emptyset$. This holds for any $R \le B_2$, proving that $[A_1, B_2]$ is a branch. $\square$

Propositions 2.4 permits us to define the maximal branch containing a given node $S \in \mathcal{T}$. Indeed, we define $\mathcal{B}_{\mathcal{T}}(S)$, the maximal branch containing S, as

$$\mathcal{B}_{\mathcal{T}}(S) = \cup\{[A, B] : [A, B] \text{ is a branch of } \mathcal{T} \text{ s.t. } S \in [A, B]\}. \tag{2.1}$$

Following [28], we say that the tree $(\mathcal{T}, \le)$ is order complete if given any totally ordered, respectively totally ordered and minorized, family $\mathcal{F} \subseteq \mathcal{T}$, the supremum, respectively the infimum, of $\mathcal{F}$ exists in $\mathcal{T}$. The supremum and the infimum of $\mathcal{F}$ will be denoted by $\sup_{\mathcal{T}} \mathcal{F}$, respectively $\inf_{\mathcal{T}} \mathcal{F}$.

2.2 Topological preliminaries

Let $\Omega \subseteq I\!R^N$ be an open set such that $\overline{\Omega}$ is homeomorphic to the unit ball of $I\!R^N$ ($N \ge 2$). Note that, in particular, $\overline{\Omega}$ is compact, connected and locally connected. Moreover, $\overline{\Omega}$ is unicoherent.

Definition 2.5. *([18, §41,X]) A topological space Z is said to be unicoherent if it is connected and for any two closed connected sets A, B in Z such that $Z = A \cup B$, $A \cap B$ is connected.*

The connected components of a set $A \subseteq I\!R^N$ will be denoted by $\mathcal{CC}(A)$. If $x \in A$, the connected component of A containing x will be denoted by $\mathrm{cc}(A, x)$, and by extension we shall write $\mathrm{cc}(A, x) = \emptyset$, if $x \notin A$. If $\emptyset \neq C \subseteq A$ and C is connected, the connected component of A containing C, denoted by $\mathrm{cc}(A, C)$, is $\mathrm{cc}(X, x)$, with $x \in C$. Sometimes we will forget the point x and write $X = \mathrm{cc}(A)$ to mean that $X \in \mathcal{CC}(A)$.

In this paper, unless explicitly stated, all topological notions are referred to $\overline{\Omega}$. In particular, ∂ denotes the boundary operator in the relative topology of $\overline{\Omega}$.

Definition 2.6. *Let $A \subseteq \overline{\Omega}$. We call holes of A in $\overline{\Omega}$ the components of $\overline{\Omega} \setminus A$. Let $p_\infty \in \overline{\Omega} \setminus A$ be a reference point, and let T be the hole of A in $\overline{\Omega}$ containing p_∞. We define the saturation of A with respect to p_∞ as the set $\overline{\Omega} \setminus T$ and we denote it by $\mathrm{Sat}(A, p_\infty)$. We shall refer to T as the external hole of A and to the other holes of A as its internal holes. By extension, if $p_\infty \in A$, by convention we define $\mathrm{Sat}(A, p_\infty) = \overline{\Omega}$. Note that $\mathrm{Sat}(A, p_\infty)$ is the union of A and its internal holes.*

The reference point p_∞ acts as a point at infinity. In all what follows, we assume that the point $p_\infty \in \overline{\Omega}$ on which the saturations are based is fixed, i.e., all saturations will be computed with respect to p_∞. To simplify our notation, we shall write $\mathrm{Sat}(A)$ instead of $\mathrm{Sat}(A, p_\infty)$. We shall also speak of holes of A instead of holes of A in $\overline{\Omega}$. Ii Figure 3 we illustrate the saturations of a set A with respect to the point at infinity p.

Definition 2.7. *Let $A \subseteq \overline{\Omega}$ and $p_\infty \in \overline{\Omega} \setminus A$. The external boundary of A, denoted by $\partial_e A$, is the boundary of the external hole of A, thus it coincides with $\partial\mathrm{Sat}(A)$. The internal boundaries of A are the boundaries of its internal holes.*

The following result summarizes the main properties of the saturation operator. Its proof can be found in [22, 2, 7]. The unicoherence of $\overline{\Omega}$ is the fundamental property that guarantees that the boundary of a hole of a connected set is connected and this implies (vi) and (ix).

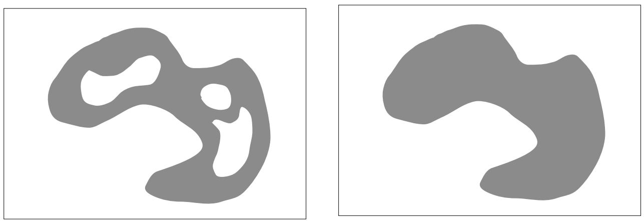

Figure 3: A set A (in light gray) and its saturation with respect to p.

Lemma 2.8. *Let $A, B \subseteq \overline{\Omega}$. The saturation operator satisfies the following properties:*

(i) *If A is open (resp. closed) in $\overline{\Omega}$, then* $\mathrm{Sat}(A)$ *is open (resp. closed) in $\overline{\Omega}$.*

(ii) *Monotonicity: If $A \subseteq B$, then* $\mathrm{Sat}(A) \subset \mathrm{Sat}(B)$

(iii) *Idempotency:* $\mathrm{Sat}\left[\mathrm{Sat}(A)\right] = \mathrm{Sat}(A)$

(iv) *Assume that A is connected and T is a hole of A. If T is an internal hole,* $\mathrm{Sat}(T) = T$*; if T is the external hole,* $\mathrm{Sat}(T) = \overline{\Omega}$.

(v) *If A is connected, then* $\mathrm{Sat}(A)$ *is also connected.*

(vi) *If A is an open (closed) connected subset of $\overline{\Omega}$, then $\partial sat(A)$ is connected.*

(vii) *If T is a hole of A, then $\partial T \subseteq \partial A$. As a consequence,* $\partial \mathrm{Sat}(A) \subseteq \partial A$.

(viii) *Assume that* $\mathrm{Sat}(A) \neq \overline{\Omega}$. *Then* $\mathrm{Sat}(A) \subseteq \mathrm{Sat}(\partial A)$*, and, if A is closed, we get* $\mathrm{Sat}(A) = \mathrm{Sat}(\partial A)$.

(ix) *Assume that A is open or closed. Let $C \in \mathcal{CC}(A)$, and $x \in \mathrm{Sat}(C) \setminus C$. Then there exists $O \in \mathcal{CC}(\overline{\Omega} \setminus A)$ such that $x \in \mathrm{Sat}(O) \subseteq \mathrm{Sat}(C)$. Moreover, if A has a finite number of connected components, and Y is an internal hole of C, then there exists $O \in \mathcal{CC}(\overline{\Omega} \setminus A)$ such that $Y = \mathrm{Sat}(O)$.*

(x) *If A is a closed set and $C \in \mathcal{CC}(A)$, then any hole of C can be expressed as a countable union of saturations of connected components of $\overline{\Omega} \setminus A$.*

(xi) *Let K_n be a decreasing sequence of continua, $K = \cap_n K_n$. Then* $\mathrm{Sat}(K) = \cap_n \mathrm{Sat}(K_n)$.

Remark 2.9. Notice that Lemma 2.8.(ii) and (viii) says that if A is closed in $\overline{\Omega}$ and does not contain p_∞, then $\mathrm{Sat}(A)$ is determined by the boundary of A. More precisely, $\mathrm{Sat}(A)$ is determined by the external boundary of A, since using (ii) and (viii) of last Lemma we have $\mathrm{Sat}(A) = \mathrm{Sat}(\mathrm{Sat}(A)) = \mathrm{Sat}(\partial \mathrm{Sat}(A)) = \mathrm{Sat}(\partial_e A)$. Now, if A is an open set in $\overline{\Omega}$ which does not contain p_∞, $\mathrm{Sat}(A)$ is also determined by its external boundary. Indeed, if $Y = \overline{\Omega} \setminus \mathrm{Sat}(A)$ is the external hole of A. Notice that Y is closed. By the previous observation $Y = \mathrm{Sat}(Y, x) = \mathrm{Sat}(\partial Y, x) = \mathrm{Sat}(\partial \mathrm{Sat}(A), x)$ where x is any point in A. Thus $\mathrm{Sat}(A) = \overline{\Omega} \setminus \mathrm{Sat}(\partial \mathrm{Sat}(A), x) = \overline{\Omega} \setminus \mathrm{Sat}(\partial_e A, x)$.

3 Upper and lower level sets, its tree structure and the of shapes an image

To fix ideas, we shall model an image u as a map from $\overline{\Omega}$ to $I\!R$ and we shall assume for the purposes of this paper that u is a continuous function in $\overline{\Omega}$. The lower and upper level sets of u

are the sets

$$[u < \lambda] \;=\; \{x \in \overline{\Omega},\, u(x) < \lambda\} \qquad [u \geq \lambda] \;=\; \{x \in \overline{\Omega},\, u(x) \geq \lambda\}\,.$$

Lower level sets are open, while upper level sets are closed. The asymmetry is justified by the structure it induces on the shapes of the image (see Theorem 3.3 below). Upper (resp. lower) level sets are an equivalent description of the image, since we can reconstruct it by the formula

$$u(x) := \sup\{\lambda \in I\!R : x \in [u \geq \lambda]\} \qquad (\text{resp. } u(x) := \inf\{\lambda \in I\!R : x \in [u < \lambda]\}).$$

Let us denote

$$\mathcal{CCULS}(u) = \{X : X \in \mathcal{CC}([u \geq \lambda]),\, \lambda \in I\!R\}$$

and

$$\mathcal{CCLLS}(u) = \{X : X \in \mathcal{CC}([u < \lambda]),\, \lambda \in I\!R\}.$$

Observe that if $\lambda \geq \mu$ and $X \in \mathcal{CC}([u \geq \lambda])$, $Y \in \mathcal{CC}([u \geq \mu])$, then either $X \cap Y = \emptyset$ or $X \subseteq Y$. Similarly, if $X \in \mathcal{CC}([u < \lambda])$, $Y \in \mathcal{CC}([u < \mu])$, then either $X \cap Y = \emptyset$ or $Y \subseteq X$. Hence, we have:

Proposition 3.1. *Both* $(\mathcal{CCULS}(u), \subseteq)$ *and* $(\mathcal{CCLLS}(u), \subseteq)$ *are trees.*

To fuse the information of both trees into a single structure, in [24, 22], the authors introduced the so-called tree of shapes of an image. In Figure 4 we display both trees of connected components of upper and lower level sets. We also display the tree of shapes to be defined in next Section.

3.1 The tree of shapes of an image

Definition 3.2. *Given an image u, we call shapes of inferior (resp. superior) type the sets*

$$\mathrm{Sat}(\mathrm{cc}([u < \mu], x)) \quad (\textit{resp. } \mathrm{Sat}(\mathrm{cc}([u \geq \lambda], x))),$$

where $\lambda, \mu \in I\!R$, $x \in \overline{\Omega}$. We call shape of u any shape of inferior or superior type. We denote by $\mathcal{S}(u)$ the shapes of u.

We note that, by Lemma 2.8.(i), shapes of superior type are closed, while shapes of inferior type are open. Notice also that, since by Lemma 2.8.(v) shapes are connected, the only shapes of both types are $\emptyset$ and $\overline{\Omega}$.

Let us recall the following result proved in [22, 2].

Theorem 3.3. *Any two shapes are either disjoint or nested. Hence,* $(\mathcal{S}(u), \subseteq)$ *is a tree.*

The tree of shapes $\mathcal{S}(u)$ was introduced as a data structure in [24] in order to operate with the level sets of u. The authors presented this data structure in the case $\overline{\Omega} \subseteq I\!R^2$, though it could be extended in principle to any dimension. For the 3D case, this has been done in [21]. We are going to give the details of this construction and justify it mathematically. For that, we shall study the maximal branches of the trees of connected components of upper and lower level sets of u and we shall fuse them to obtain the tree of shapes of u. An example of the tree of shapes is illustrated in Figure 4.

Observe that, having fixed the point p_∞ on which the saturation is based, the shapes of the image depend on p_∞. But, as it has been shown in [22, 2], the tree of shapes of u encodes an information which does not depend on p_∞, namely the external boundaries of the shapes of the image.

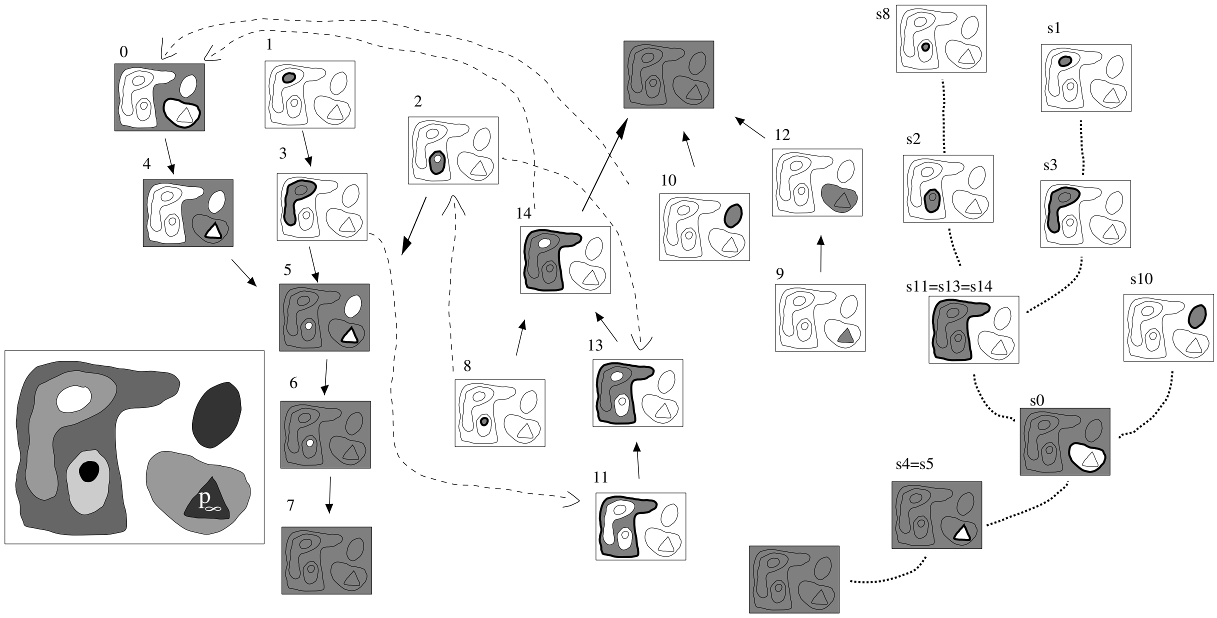

Figure 4: From left to right: a synthetic 2D image, its upper tree (with arrows down), its lower tree (with arrows up), and the set of shapes, which are saturations of upper and lower regions. To define the saturation we have used a point p_∞ inside the small triangular region. The saturation of each region is defined by its external boundary, marked in bold. Notice that the shapes have a tree structure, indicated by dotted lines. The dashed arrows show some of the hole relation between connected components of the upper and lower trees. The dotted lines of the tree of shapes come from either arrows of the original tree, or from the dashed arrows that define the external boundaries (see Section 7).

Remark 3.4. Shapes could be defined for upper semicontinuous functions and Theorem 3.3 holds also in that case. Since the results in Section 5 are proved for continuous functions we shall restrict the study of the tree to this case. To apply these results to discrete images, one has to interpolate them [22, 11].

4 The order completion of the trees

In order to study the branch structure of the trees $\mathcal{CCULS}(u)$, $\mathcal{CCLLS}(u)$, we want to express its maximal branches, defined as unions of intervals, as intervals but we cannot do this because they are not order complete. For that, let us compute its order completion (we refer to [28] for definitions) in the order complete Boolean algebra $(\mathcal{P}(\overline{\Omega}), \subseteq)$ of all subsets of $\overline{\Omega}$.

Form now on, let $\mathcal{T}$ denote any of the trees $\mathcal{CCULS}(u)$, $\mathcal{CCLLS}(u)$, or $\mathcal{S}(u)$.

Definition 4.1. *We define a limit node of $\mathcal{T}$ as the supremum, or the infimum, in $(\mathcal{P}(\overline{\Omega}), \subseteq)$ of a totally ordered family (assuming it minorized in the infimum case) of nodes of $\mathcal{T}$. The limit nodes of the tree $\mathcal{S}(u)$ will be called limit shapes of u.*

Observe that any node is a limit node. The order completion of $\mathcal{T}$, denoted by $\mathcal{T}^c$ is formed by the limit nodes of $\mathcal{T}$.

4.1 The order completion of the upper and lower trees

Proposition 4.2. *(i) If X is a limit node of $\mathcal{CCULS}(u)$, then either $X \in \mathcal{CCULS}(u)$ or $X \in \mathcal{CC}([u > \lambda]$ for some $\lambda \in I\!R$. Thus, the order completion of the tree $\mathcal{CCULS}(u)$, is given by*

$$\mathcal{CCULS}^c(u) := \mathcal{CCULS}(u) \cup \{X : X \in \mathcal{CC}([u > \lambda]), \lambda \in I\!R\}.$$

(ii) If X is a limit node of $\mathcal{CCLLS}(u)$, then either $X \in \mathcal{CCLLS}(u)$ or $X \in \mathcal{CC}([u \leq \lambda]$ for some $\lambda \in I\!R$. Thus, the order completion of the tree $\mathcal{CCLLS}(u)$, is given by

$$\mathcal{CCLLS}^c(u) := \mathcal{CCLLS}(u) \cup \{X : X \in \mathcal{CC}([u \leq \lambda]), \lambda \in I\!R\}.$$

Moreover, for both trees, any two limit nodes are nested or disjoint. Hence, the completions are also trees.

Proof. Being identical, we just give the proof of (i). If X is a limit node of $\mathcal{CCULS}(u)$, then X is an inf or a sup of an ordered set of upper connected components which we may assume countable. If $X = \cap_n X_n$ where $X_n \in \mathcal{CC}([u \geq \lambda_n])$ with $\lambda_n \uparrow \lambda$, then $X \in \mathcal{CC}([u \geq \lambda])$. To prove it, let $x \in X$ and $u(x) \geq \lambda$. Then $X_n = \mathrm{cc}([u \geq \lambda_n], x)$ and $\mathrm{cc}([u \geq \lambda], x) \subseteq \cap_n X_n = X$. Now, since X_n is a decreasing sequence of continua, we know that X is a continuum contained in $[u \geq \lambda]$ [18]. This implies that $X \subseteq \mathrm{cc}([u \geq \lambda], x)$.

If $X = \cup_n X_n$ where $X_n \in \mathcal{CC}([u \geq \lambda_n])$ with $\lambda_n \downarrow \lambda$, then $X \in \mathcal{CC}([u > \lambda])$. To prove this, let $x \in X_1$. Observe that $X_n = \mathrm{cc}([u > \lambda_n], x)$. For any n, $X_n \subseteq [u > \lambda]$, thus $\bigcup_{n \in I\!N} X_n \subseteq \mathrm{cc}([u > \lambda], x)$. On the other hand, $[u > \lambda]$ being open and $\overline{\Omega}$ locally connected, $\mathrm{cc}([u > \lambda], x)$ is an open set. Hence, for any $y \in \mathrm{cc}([u > \lambda], x)$, there is some continuum $K_y \subseteq \mathrm{cc}([u > \lambda], x)$ containing x and y. Since $K_y \subseteq \bigcup_{n \in I\!N} [u > \lambda_n]$ and it is a compact set, we can extract a finite covering of K_y, and as the sequence $[u > \lambda_n]$ is nondecreasing, there is some n such that $K_y \subseteq [u > \lambda_n]$. Since K_y is connected and contains x, we have that $y \in K_y \subseteq \mathrm{cc}([u > \lambda_n], x) = X_n$. We conclude that $\mathrm{cc}([u > \lambda], x) \subseteq \bigcup_{n \in I\!N} X_n$.

The proof of last statement is straightforward and we do not give the details. □

Thanks to Proposition 4.2 we may compute the maximal branch containing a given node $S \in \mathcal{T}$. If $\mathcal{T} = \mathcal{CCULS}(u)$, then $\inf_{\mathcal{T}^c} \mathcal{B}_\mathcal{T}(S) \in \mathcal{CCULS}(u)$ and we may compute $\sup_{\mathcal{T}^c} \mathcal{B}_\mathcal{T}(S) \in \mathcal{T}^c$. If $\mathcal{T} = \mathcal{CCLLS}(u)$, then $\sup_{\mathcal{T}^c} \mathcal{B}_\mathcal{T}(S) \in \mathcal{CCLLS}(u)$ and we may compute $\inf_{\mathcal{T}^c} \mathcal{B}_\mathcal{T}(S) \in \mathcal{T}^c$. Hence, we may write

$$\mathcal{B}_\mathcal{T}(S) = [\inf_{\mathcal{T}^c} \mathcal{B}_\mathcal{T}(S), \sup_{\mathcal{T}^c} \mathcal{B}_\mathcal{T}(S)]_{\mathcal{T}^c} \cap \mathcal{T}.$$

If there is no confusion we shall write a maximal branch as $\mathcal{B} = [A, B]$ with the implicit understanding that $A, B \in \mathcal{T}^c$ and $[A, B]$ denotes $[A, B]_{\mathcal{T}^c} \cap \mathcal{T}$. Under some assumption on u, a more detailed description of maximal branches will be given in Propositions 5.7 and 5.8. An example of the maximal branches of an image is given in Figure 5.

4.2 The order completion of the tree of shapes for weakly oscillating functions

Definition 4.3. *Let $u \in C(\overline{\Omega})$ and $M \subseteq \overline{\Omega}$. We say that M is a regional maximum (resp., minimum) of u at height λ if M is a connected component of $[u = \lambda]$ and, for all $\varepsilon > 0$, the set $[\lambda - \varepsilon < u \leq \lambda]$ (resp., $[\lambda \leq u < \lambda + \varepsilon]$) is a neighborhood of M. The regional extrema of u are the regional maxima and minima of u*

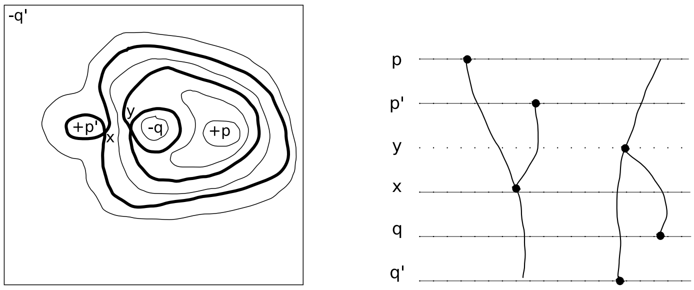

Figure 5: A function and its upper and lower trees with its maximal branches.

Definition 4.4. *We say that $u \in C(\overline{\Omega})$ is weakly oscillating if it has a finite number of regional extrema.*

The following proposition was proved in [2, 8] and it characterizes the limit shapes of u. Its proof uses both Lemma 2.8.(ix) and (x) and Lemma 5.2 below. We shall not include the proof here.

Proposition 4.5. *Let $u \in C(\overline{\Omega})$ be a weakly oscillating function. Then the limit shapes of u are sets of the form $\mathrm{Sat}(C)$ where either $C \in \mathcal{CC}([u \geq \lambda])$, or $C \in \mathcal{CC}([u > \lambda])$, or $C \in \mathcal{CC}([u \leq \lambda])$, or $C \in \mathcal{CC}([u < \lambda])$. Moreover, any two limit nodes of $\mathcal{S}(u)$ are either nested or disjoint. Thus, the order completion of $\mathcal{S}(u)$, denoted by $\mathcal{S}^c(u)$ is also a tree.*

5 Weakly oscillating functions and the structure of its trees

Definition 5.1. *We call leaf of the tree $\mathcal{T}$, or simply, a leaf, any limit node $L = \inf_{\mathcal{T}^c}[A, B]$ containing no other node of $\mathcal{T}$.*

Leaves are, thus, minimal elements in $\mathcal{T}^c$. Our purpose is to describe the leaves and maximal branches of the upper and lower trees.

Lemma 5.2. *Let $u \in C(\overline{\Omega})$ be a weakly oscillating function. Then for each $\lambda \in I\!R$, if $X \in \mathcal{CC}([u > \lambda])$ or $X \in \mathcal{CC}([u \geq \lambda])$ is nonempty, then X contains a regional maximum of u. A similar statement holds for lower level sets. Thus, for each $\lambda \in I\!R$, there is a finite number of connected components of $[u \geq \lambda]$ and each component has a finite number of holes.*

Proof. Let $X \in \mathcal{CC}([u > \lambda])$ be nonempty. Then $\mu := \max_{x \in X} u(x) > \lambda$ is attained at a point $p \in X$. Let $Y = \mathrm{cc}([u = \mu], p)$. Observe that $Y \subseteq X$. On the other hand, $u \leq \mu$ near Y, since otherwise we would find a point $q \in X$ with $u(q) > \mu$. We conclude that for all $\varepsilon > 0$ the set $[\mu - \varepsilon < u \leq \mu]$ is a neighborhood of Y, hence Y is a regional extremum of u. In particular, the number of connected components of $[u > \lambda]$ is finite.

Let us prove that each connected component of $[u \geq \lambda]$ contains a regional extremum of u. By the previous paragraph, we know that the connected components of $[u \geq \lambda]$ which intersect $[u > \lambda]$ contain a regional extremum and are finite in number. We denote them by $X_1, \ldots, X_k$. Let $X \in \mathcal{CC}([u \geq \lambda])$ be such that $X \subseteq [u = \lambda]$. Let us prove that X is a regional extremum of u. Obviously, X is a connected component of $[u = \lambda]$. Let $\eta > 0$ be such that $d(\cup_{i=1}^k X_i, X) \geq \eta$. We have that for all $\varepsilon > 0$ the set $[\lambda - \varepsilon < u \leq \lambda]$ is a neighborhood of X. Otherwise, there exists a sequence $p_n \to p \in X$ such that $u(p_n) > \lambda$. Then for each n, $p_n \in \cup_{i=1}^k X_i$ and, thus, $d(p_n, X) \geq \eta$, a contradiction since p_n converges to a point in X. We conclude that any connected component of $[u \geq \lambda]$ contains a regional maximum, and, thus, there must be a finite number of them.

The corresponding statements for lower level sets follow from the previous ones applied to $-u$.

Let $\lambda \in I\!R$. Let X be a connected component of $[u \geq \lambda]$ and let H be a hole of X. Observe that $\partial H \subseteq \partial X \subseteq X$. Since $X \cap \overline{H} \neq \emptyset$ and $X, \overline{H}$ are connected, then $X \cup H = X \cup \overline{H}$ is connected. If $H \subseteq [u \geq \lambda]$, then $H \subseteq X$, a contradiction. Hence $H \cap [u < \lambda] \neq \emptyset$. We conclude that each hole of X contains a component of $[u < \lambda]$. Hence there may be only a finite number of them. $\square$

Lemma 5.3. *Let $u \in C(\overline{\Omega})$ be a weakly oscillating function. Let $X \in \mathcal{CC}([\lambda \leq u \leq \mu])$, $\lambda \leq \mu$, and let H be a hole of X. Then H is the saturation of a connected component either of $[u < \lambda]$ or of $[u > \mu]$.*

Proof. By Lemma 2.8.(x) there exist a sequence of connected components $\{O_n\}_n$ of $[u < \lambda] \cup [u > \mu]$ such that $\mathrm{Sat}(O_n)$ are increasing and $H = \cup_n \mathrm{Sat}(O_n)$. Observe that O_n are two by two disjoint. Without loss of generality we may assume that O_n are all connected components of $[u < \lambda]$. Thus, by Lemma 5.2, there are only finitely many of them, and there is a set $O \in \mathcal{CC}([u < \lambda])$ such that $H = \mathrm{Sat}(O)$. $\square$

Lemma 5.4. *Let $u \in C(\overline{\Omega})$ be a weakly oscillating function. Let X be a connected component of $[\lambda \leq u \leq \mu]$, $\lambda \leq \mu$, and let L be a hole of X. Then there is some $\eta > 0$ such that either*

i) $\mathrm{Sat}(X, L) = \mathrm{Sat}(\mathrm{cc}([u \geq \lambda], X), L)$, *and $u < \lambda$ on $L_\eta := \{p \in L : d(p, X) < \eta\}$, or*

ii) $\mathrm{Sat}(X, L) = \mathrm{Sat}(\mathrm{cc}([u \leq \mu], X), L)$, *and $u > \mu$ on $L_\eta := \{p \in L : d(p, X) < \eta\}$.*

In the first case of the alternative holds, we say that L is a hole of negative type, in the second case we say that L is a hole of positive type.

Proof. We may assume that $L \neq \emptyset$, otherwise all saturations in the statement are equal to $\overline{\Omega}$ and the result is true. Assume that $\lambda < \mu$. By Lemma 5.3, we may write $L = \mathrm{Sat}(O)$ where either $O \in \mathcal{CC}([u < \lambda]$, or $O \in \mathcal{CC}([u > \mu])$. To fix ideas, assume that $O \in \mathcal{CC}([u < \lambda]$ (in particular, this implies that $[u < \lambda] \neq \emptyset$). Then $\partial L \subseteq \partial O \subseteq \partial[u < \lambda] \subseteq [u = \lambda]$. Let us prove that, for some $\eta > 0$, $u < \lambda$ on L_η.

Let us prove that the connected components of $[u \geq \lambda]$ are either disjoint to L, or contained in L. Let Y be a connected component of $[u \geq \lambda]$ intersecting L. Then $Y \subseteq L$. Otherwise, let $p \in Y \cap L$, $q \in Y \setminus L$, and let K be a continuum containing p and q and contained in Y. In this case, we have that $K \cap O \neq \emptyset$, a contradiction since $K \subseteq [u \geq \lambda]$. Indeed, if $K \cap O = \emptyset$, then K is contained in a hole of O. Since $\overline{\Omega} \setminus L$ is a hole of O containing $q \in K$, we have that $K \subseteq \overline{\Omega} \setminus L$, and this is a contradiction since $p \in K \cap L$.

Observe that if $Y \in \mathcal{CC}([u \geq \lambda])$, $Y \subseteq L$, then $\partial Y \cap \partial L = \emptyset$. Indeed, on one hand, we have $\mathrm{dist}(Y, X) > 0$ (otherwise, if this distance is null, then $X \cap Y \neq \emptyset$, hence $X \subseteq Y \subseteq L$, a contradiction). This implies that $\partial Y \cap \partial X = \emptyset$. On the other hand, we have $\partial L \subseteq \partial X$. Hence, $\partial Y \cap \partial L = \emptyset$. Since, by Lemma 5.2, $[u \geq \lambda]$ has a finite number of connected components, we deduce that $\mathrm{dist}([u \geq \lambda] \cap L, \partial L) > 0$, and, therefore, we have $L_\eta \subseteq [u < \lambda]$ for some $\eta > 0$. This implies that L is a hole of $\mathrm{cc}([u \geq \lambda], X)$, and $\mathrm{Sat}(X, L) = \mathrm{Sat}(\mathrm{cc}([u \geq \lambda], X), L)$.

Let us consider the case $\lambda = \mu$. By assumption X is a connected component of $[u = \lambda]$ and L is a hole of X. Let $y \in X$. Then $X = \cap_n X_n$ where $X_n = \mathrm{cc}([\lambda \leq u \leq \lambda + \frac{1}{n}], y)$. Let $p \in L$. Then, by Lemma 2.8.(xi), we know that $\mathrm{Sat}(X, p) = \cap_n \mathrm{Sat}(X_n, p)$. Without loss of generality, we may assume that $p \notin X_n$ for all $n \geq 1$. But, according to the first part of the proof, we have that either $\mathrm{Sat}(X_n, p) = \mathrm{Sat}(\mathrm{cc}([u \geq \lambda], y), p)$, or $\mathrm{Sat}(X_n, p) = \mathrm{Sat}(\mathrm{cc}([u \leq \lambda + \frac{1}{n}], y), p)$. In the first case, we conclude that $\mathrm{Sat}(X, p) = \mathrm{Sat}(\mathrm{cc}([u \geq \lambda], y), p)$. In the second case, using again Lemma 2.8.(xi), we have that $\cap_n \mathrm{Sat}(\mathrm{cc}([u \leq \lambda + \frac{1}{n}], y), p) = \mathrm{Sat}(\mathrm{cc}([u \leq \lambda], y), p)$. Hence, $\mathrm{Sat}(X, p) = \mathrm{Sat}(\mathrm{cc}([u \leq \lambda], y), p)$.

When $\mathrm{Sat}(X, p) = \mathrm{Sat}(\mathrm{cc}([u \geq \lambda], y), p)$, L is a hole of $\mathrm{cc}([u \geq \lambda], y)$. Hence $\partial L \subseteq \partial[u < \lambda]$ and the argument above proves that there is some $\eta > 0$ such that $u < \lambda$ on $L_\eta = \{p \in L : d(p, X) < \eta\}$. When $\mathrm{Sat}(X, p) = \mathrm{Sat}(\mathrm{cc}([u \leq \lambda], y), p)$, L is a hole of $\mathrm{cc}([u \leq \lambda], y)$. Then $\partial L \subseteq \partial[u > \lambda]$ and again the previous argument proves that there is some $\eta > 0$ such that $u > \lambda$ on $L_\eta = \{p \in L : d(p, X) < \eta\}$. $\square$

Proposition 5.5. *Let $u \in C(\overline{\Omega})$ be a weakly oscillating function. Then*

(i) If X is a leaf of the tree $\mathcal{CCULS}(u)$, then X is a regional maximum of u and $X = \mathrm{cc}([u = \lambda])$ for some $\lambda \in I\!R$.

(ii) If X is a leaf of the tree $\mathcal{CCLLS}(u)$, then X is a regional minimum of u and $X = \mathrm{cc}([u = \lambda])$ for some $\lambda \in I\!R$.

(iii) If X is a leaf of the tree $\mathcal{S}(u)$, then X is a regional extremum of u and $X = \mathrm{cc}([u = \lambda])$ for some $\lambda \in I\!R$.

Proof. (i) By Proposition 4.2, if X is a leaf of the tree $\mathcal{CCULS}(u)$, then $X \in \mathcal{CC}([u \geq \lambda])$ for some $\lambda \in I\!R$. If $u(x) > \lambda$ for some $x \in X$, then the node $\mathrm{cc}([u \geq u(x)], x)$ is nonempty and contained in X. Thus $u = \lambda$ on X and $X \in \mathcal{CC}([u = \lambda])$. If $X = \overline{\Omega}$, our statement is obviously true. If $X \neq \overline{\Omega}$, then by Lemma 5.4 all holes of X must be of negative type. Hence $\mathrm{cc}([u > \lambda - \epsilon], X) = \mathrm{cc}([\lambda - \epsilon < u \leq \lambda], X)$, for any $\epsilon > 0$, and $\mathrm{cc}([u > \lambda - \epsilon], X)$ is an open set containing X.

Being similar to the proof of (i), we skip the proof of (ii).

(iii) Assume that X is a leaf of $\mathcal{S}(u)$. Then, by Proposition 4.5, then $X = \mathrm{Sat}(Y)$ where either a) $Y = \mathrm{cc}([u \geq \lambda])$ or b) $Y = \mathrm{cc}([u \leq \lambda])$ for some $\lambda \in I\!R$. If we are in case a), then we may argue as in (i). If we are in case b) we may argue as in (ii). We conclude that X is a regional extremum of u. $\square$

Lemma 5.6. *Let $u \in C(\overline{\Omega})$ be a weakly oscillating function. Let $\lambda \in I\!R$ and let $X^{\lambda,i}$, $X_{\lambda,j}$, $i = 1, ..., r$, $j = 1, ..., s$, be the the family of connected components of $[u \geq \lambda]$, resp. $[u < \lambda]$. There is $\varepsilon > 0$ such that for any $\mu \in (\lambda - \varepsilon, \lambda]$, there are exactly r connected components $X^{\mu,i}$, $i = 1, \ldots, r$ of $[u \geq \mu]$, where $X^{\mu,i}$ contains $X^{\lambda,i}$ and $X^{\lambda,i} = \cap_{\mu<\lambda} X^{\mu,i}$. Moreover each $X^{\mu,i}$ contains the same family of regional extrema as $X^{\lambda,i}$, $i = 1, \ldots, r$. There are also s connected components $X_{\mu,j}$, $j = 1, \ldots, s$, of $[u < \mu]$, where $X_{\mu,j}$ is contained in $X_{\lambda,j}$ and $X_{\lambda,j} = \cup_{\mu<\lambda} X_{\mu,j}$. Moreover each $X_{\mu,j}$ contains the same family of regional extrema as $X_{\lambda,j}$, $j = 1, \ldots, s$.*

Proof. Let $i \in \{1, ..., r\}$. For each $\mu < \lambda$, let $X^{\mu,i}$ be the connected component of $[u \geq \mu]$ containing $X^{\lambda,i}$. Then, obviously, we have

$$X^{\lambda,i} \subseteq \cap_{\mu<\lambda} X^{\mu,i}.$$

Now, since $X^{\mu,i}$ is a decreasing sequence of continua their intersection is also a continuum [18]. Moreover, it is contained in $[u \geq \lambda]$. Therefore,

$$\cap_{\mu<\lambda} X^{\mu,i} \subseteq \mathrm{cc}([u \geq \lambda], p_i) = X^{\lambda,i},$$

and we have the equality of both sets. As a consequence, there is an $\varepsilon > 0$ such that for each $\mu \in (\lambda - \varepsilon, \lambda]$, the sets $X^{\lambda,i}$, $i = 1, ..., r$, are contained in different connected components of $[u \geq \mu]$. Moreover, since the number of connected components of each $[u \geq \mu]$ is finite, we may choose $\varepsilon > 0$ such that for each $\mu \in (\lambda - \varepsilon, \lambda]$ the set $[u \geq \mu]$ consists of r connected components, each one of them containing a different component of $[u \geq \lambda]$. Since u is weakly oscillating, for $\epsilon > 0$ small enough, the regional extrema of u in each $X^{\mu,i}$, $i = 1, \ldots, r$, is constant for $\mu \in (\lambda - \epsilon, \lambda]$.

Again, using that $\cup_{\mu<\lambda}[u < \mu] = [u < \lambda]$, for $\epsilon > 0$ small enough and $\mu \in (\lambda - \epsilon, \lambda)$, we have that $[u < \mu] \cap X_{\lambda,j}$, $j = 1, ..., s$, are the connected components of $[u < \mu]$. As above, we know that the regional extrema of u in each $[u < \mu] \cap X_{\lambda,j}$ coincide with the regional extrema in $X_{\lambda,j}$, $j = 1, \ldots, s$, for $\epsilon > 0$ small enough and $\mu \in (\lambda - \epsilon, \lambda)$. □

Proposition 5.7. *Assume that $u \in C(\overline{\Omega})$ is a weakly oscillating function. The tree $\mathcal{CCULS}(u)$ has a finite number of leaves and a finite number of maximal branches. If $\mathcal{B} = [A, B]$ is a maximal branch of $\mathcal{CCULS}(u)$ with A, B being limit nodes, then*

a) either $B = \mathrm{cc}([u \geq \lambda]) = \overline{\Omega}$ or $B \in \mathcal{CC}([u > \lambda])$ for some $\lambda \in I\!R$, there is no bifurcation between A and B, and if $B' = \mathrm{cc}([u \geq \lambda], B)$, then $[A, B']$ contains a bifurcation. We have $B' = \inf[B, \overline{\Omega}]$. In this case, we call B (resp. B'), a bifurcating limit node (resp. a bifurcating node) at level λ. Moreover, B cannot be a leaf unless u is constant.

b) $A \in \mathcal{CC}([u \geq \lambda])$ for some $\lambda \in I\!R$ and either A is a leaf, or for any $X \in \mathcal{CCULS}(u)$, $X \subsetneq A$, $[X, A]$ contains a bifurcation. In this second case, we call it a bifurcating node (at level λ).

Proof. Leaves of $\mathcal{CCULS}(u)$ are regional maxima of u, hence there are finitely many of them. Let $\mathcal{B} = [A, B]$ be a maximal branch in $\mathcal{CCULS}(u)$. By Proposition 4.2 either $B \in \mathcal{CC}([u \geq \lambda])$ or $B \in \mathcal{CC}([u > \lambda])$ for some $\lambda \in I\!R$. If $B \in \mathcal{CC}([u \geq \lambda])$ and $\lambda > \inf_{x\in\overline{\Omega}} u(x)$, then, using Lemma 5.6, we would be able to extend the branch $\mathcal{B}$ to the right. Hence, $\lambda = \inf_{x\in\overline{\Omega}} u(x)$, i.e. $B = \overline{\Omega}$. If $B \in \mathcal{CC}([u > \lambda])$, then $[A, B]$ does not contain a bifurcation. Let $B' = \mathrm{cc}([u \geq \lambda], B)$, then $[A, B']$ must contain a bifurcation, otherwise $\mathcal{B}$ would not be maximal. The argument in Lemma 5.6 proves that $B' = \inf[B, \overline{\Omega}]$. The last assertion follows from Proposition 5.5.

Now, observe that $A \in \mathcal{CC}([u \geq \lambda])$ for some $\lambda \in I\!R$. Since $\mathcal{B}$ is maximal, if A is not a leaf, then for any $X \in \mathcal{CCULS}(u)$, $X \subsetneq A$, $[X, A]$ contains a bifurcation.

Since u has a finite number of regional maxima, there are finitely many maximal branches in $\mathcal{CCULS}(u)$, since any two of them are disjoint. □

Let us prove the corresponding result for $\mathcal{CCLLS}(u)$.

Proposition 5.8. *Assume that $u \in C(\overline{\Omega})$ is a weakly oscillating function. The tree $\mathcal{CCLLS}(u)$ has a finite number of leaves and a finite number of maximal branches. If $\mathcal{B} = [A, B]$ is a maximal branch of $\mathcal{CCLLS}(u)$ with A, B being limit nodes, then*

a) either $B = \mathrm{cc}([u \leq \lambda]) = \overline{\Omega}$ or $B \in \mathcal{CC}([u < \lambda])$ for some $\lambda \in I\!R$, there is no bifurcation between A and B, and if $B' = \mathrm{cc}([u \leq \lambda], B)$, then $[A, B']$ contains a bifurcation. We have $B' = \inf(B, \overline{\Omega}]$ where $(B, \overline{\Omega}] = [B, \overline{\Omega}] \setminus \{B\}$. In this case, we call B (resp. B'), a bifurcating node (resp. a bifurcating limit node) at level λ. Moreover, B cannot be a leaf unless u is constant.

b) $A \in \mathcal{CC}([u \leq \lambda])$ *for some* $\lambda \in \mathbb{R}$ *and either* A *is a leaf, or for any* $X \in \mathcal{CCLLS}(u)$, $X \subsetneq A$, $[X, A]$ *contains a bifurcation, indeed it contains at least two connected components of* $[u < \lambda]$. *In this second case, we call it a bifurcating node (at level* λ*).*

For simplicity, we shall say simply bifurcating node independently of being a node or a limit node.

Proof. Leaves are regional minima of u, hence there are finitely many of them. Let $\mathcal{B} = [A, B]$ be a maximal branch in $\mathcal{CCLLS}(u)$. By Proposition 4.2 either $B \in \mathcal{CC}([u \leq \lambda])$ or $B \in \mathcal{CC}([u < \lambda])$ for some $\lambda \in \mathbb{R}$. If $B \in \mathcal{CC}([u \leq \lambda])$ and $\lambda < \sup_{x \in \overline{\Omega}} u(x)$, then, using the arguments in Lemma 5.6, we would be able to extend the branch $\mathcal{B}$ to the right. Hence, $\lambda = \sup_{x \in \overline{\Omega}} u(x)$ and $B = \overline{\Omega}$.

If $B \in \mathcal{CC}([u < \lambda])$, then $[A, B]$ does not contain a bifurcation. Let us prove that $[A, B']$ contains a bifurcation where $B' = \mathrm{cc}([u \leq \lambda], B)$. If $B' = \overline{\Omega}$ and it does not contain a bifurcation we are in the previous case for B (we could take $B = B'$). Thus, we may assume that $B' = \overline{\Omega}$ and $[A, B']$ contains a bifurcation; or $B' \neq \overline{\Omega}$, that is $\lambda < \sup_{x \in \overline{\Omega}} u(x)$. Let us consider this last case. By (the proof of) Lemma 5.6, there is an $\epsilon > 0$ such that if $\mu \in (\lambda, \lambda + \epsilon)$, then $\mathrm{cc}([u < \mu], B')$ does not contain any other connected component of $[u \leq \lambda]$ besides B' and contains the same regional extrema as B'. Let $C = \mathrm{cc}([u < \mu], B')$ with $\lambda < \mu < \lambda + \epsilon$. It $[B, C]$ does not bifurcate, we would be able to extend $[A, B]$ to the right. Thus, we may assume that $[B, C]$ contains a bifurcation, i.e., there is $Y \in \mathcal{CC}([u < \alpha])$ with $Y \cap B = \emptyset$ and $Y \subseteq C$. Notice that we have $\alpha \leq \mu$. If $\lambda < \alpha \leq \mu$ and $Y \cap [u \leq \lambda] \neq \emptyset$, then we consider V as the connected component of $[u \leq \lambda]$ inside Y. Then C contains V and B', but this is not possible in view of Lemma 5.6. If $Y \subseteq [\lambda < u \leq \mu]$, then Y contains a regional minimum not in B', hence also C does it, a contradiction with Lemma 5.6. Thus, we may assume that $\alpha \leq \lambda$. Let $V = \mathrm{cc}([u < \lambda], Y)$. Since $Y \cap B = \emptyset$, we have that $V \cap B = \emptyset$. If $\mathrm{cc}([u \leq \lambda], V)$ is disjoint to B', then C contains two connected components of $[u \leq \lambda]$, a contradiction with Lemma 5.6. Otherwise, $B' = \mathrm{cc}([u \leq \lambda], V)$, and in this case $[A, B']$ contains a bifurcation since B and V are disjoint. Finally, the argument in Lemma 5.6 proves that $B' = \inf(B, \overline{\Omega}]$. The last assertion in a) follows from Proposition 5.5.

Since A is a limit node, then $A \in \mathcal{CC}([u \leq \lambda])$ for some $\lambda \in \mathbb{R}$. Let us prove that if A is not a leaf, then for any $X \in \mathcal{CCLLS}(u)$, $X \subsetneq A$, $[X, A]$ contains a bifurcation, i.e., there is $Y \in \mathcal{CCLLS}(u)$, $Y \subseteq A$ and $Y \cap X = \emptyset$. Observe that $\inf_A u < \lambda$, otherwise $A \in \mathcal{CC}([u = \lambda])$ and A is a leaf. If there are more than one connected component of $[u < \lambda]$ in A, our assertion is true. Thus, we may assume that there is only one connected component of $[u < \lambda]$ in A. Let $S \in [A, B]$, $S \in \mathcal{CC}([u < \lambda'])$ with $\lambda < \lambda'$. Observe that $[X, S]$ contains a bifurcation otherwise we could enlarge $[A, B]$ to the left. Let $Y \in \mathcal{CCLLS}(u)$ be such that $Y \subseteq S$ and $Y \cap X = \emptyset$. Notice that we may write that $X \in \mathcal{CC}([u < \alpha])$ with $\alpha \leq \lambda$ and $Y \in \mathcal{CC}([u < \mu])$ for some $\mu < \lambda'$ (if $\mu = \lambda'$, then $Y = S$, contradicting the fact that $X \cap Y = \emptyset$). Since there is only one connected component of $[u < \lambda]$ in S (otherwise there would be a bifurcation in $[A, S]$ since A identifies one of them), we have that $Y \subseteq A$ or $Y \subseteq [\lambda \leq u < \mu]$. In the first case, we have that there is a bifurcation in $[X, A]$. Let us consider the second case: if $Y \cap A = \emptyset$, then $[A, B]$ contains a bifurcation. Hence $Y \cap A \neq \emptyset$. Since Y is a node and A a limit node, then either $A \subseteq Y$, a contradiction since $X \subseteq A$ and $X \cap Y = \emptyset$, or $Y \subseteq A$, in which case $Y \in \mathcal{CC}([u = \lambda])$, $Y = A$ and A would be a leaf. We conclude that there are more than one connected component of $[u < \lambda]$ in A and $[X, A]$ contains a bifurcation.

Since u has a finite number of regional maxima, there are finitely many maximal branches in $\mathcal{CCLLS}(u)$, since any two of them are disjoint. $\square$

6 Signatures and singular values of the trees

We may interpret the branch structure of $\mathcal{CCULS}(u)$ and $\mathcal{CCLLS}(u)$ in terms of the singularities of both trees. For that, let us describe the concepts of critical values of the upper and lower trees.

Definition 6.1. *For $X \in \mathcal{CC}([u \geq \lambda])$, we call upper signature of X and note $sig^+(X)$ the set $sig^+(X) = \{E : E$ is a regional maximum, $E \subseteq X\}$. If $[u \geq \lambda]$ is an upper level set of u, we define its signature as $sig^+([u \geq \lambda]) = \{sig^+(C) | C \in \mathcal{CC}([u \geq \lambda])\}$. We write $sig^+(\lambda) = sig([u \geq \lambda])$.*

For $X \in \mathcal{CC}([u < \lambda])$, we call lower signature of X and note $sig^-(X)$ the set $sig^-(X) = \{E : E$ is a regional minimum, $E \subseteq X\}$. If $[u < \lambda]$ is a lower level set of u, we define its signature as $sig^-([u < \lambda]) = \{sig^-(C) | C \in \mathcal{CC}([u < \lambda])\}$. We write $sig^-(\lambda) = sig([u < \lambda])$.

The following Lemma is a consequence of Lemma 5.6:

Lemma 6.2. *Let $u \in C(\overline{\Omega})$ be a weakly oscillating function. Let $\lambda \in I\!R$. There is $\varepsilon > 0$ such that $sig^+(\mu)$ and $sig^-(\mu)$ are constant for all $\mu \in (\lambda - \varepsilon, \lambda]$.*

Definition 6.3. *Let $u \in C(\overline{\Omega})$ be a weakly oscillating function. We say that $\lambda \in I\!R$ is a critical value of the upper (resp. lower) tree of u if there is a sequence $\mu_n \downarrow \lambda$ such that $sig^+([u \geq \mu_n]) \neq sig^+([u \geq \lambda])$ (resp. $sig^-([u \geq \mu_n]) \neq sig^-([u \geq \lambda])$) for each $n = 1, 2, ...$ We denote the criticalities of the upper (resp. lower) tree by $\mathcal{CUT}(u)$ (resp. $\mathcal{CLT}(u)$).*

Since as λ decreases, $sig^+(\lambda)$ increases, resp. $sig^-(\lambda)$ decreases, there are only finitely many possible changes in sig^+ and sig^-. Thus we have:

Proposition 6.4. *Let $u \in C(\overline{\Omega})$ be a weakly oscillating function. The number of critical values of the upper and lower trees of u is finite.*

Hence if λ is a critical value of $\mathcal{CCULS}(u)$ (resp. of $\mathcal{CCLLS}(u)$), then there is an $\epsilon > 0$ such that $sig^+(\mu)$ (resp. $sig^-(\mu)$) is constant and different from $sig^+(\lambda)$ (resp. $sig^-(\lambda)$) for any $\mu \in (\lambda, \lambda + \epsilon)$ and $sig^+(\mu) = sig^+(\lambda)$ (resp. $sig^-(\mu) = sig^-(\lambda)$) for any $\mu \in (\lambda - \epsilon, \lambda]$.

Proposition 6.5. *We have that $\lambda \in \mathcal{CUT}(u)$ (resp. $\lambda \in \mathcal{CLT}(u)$) if and only if there is either a leaf or a bifurcating node of $\mathcal{CCULS}(u)$ (resp. $\mathcal{CCLLS}(u)$) at level λ.*

In other words, the signature is constant along maximal branches.

Proof. Since $sig^+(\mathrm{cc}([u \geq \lambda]))$ increases as λ decreases, the only possible changes in $sig^+(\lambda)$ as λ decreases reflect the following facts: a) the birth of a new connected component of $[u \geq \lambda]$ at level λ, or b) two different connected components of $[u \geq \mu]$ for $\mu \in (\lambda, \lambda + \epsilon)$ merged at level λ. In the first case, there is a leaf of $\mathcal{CCULS}(u)$ at level λ. In the second case, we have a bifurcating node $\mathcal{CCULS}(u)$ at level λ. The converse assertion is clear from Proposition 5.5. The assertions for $\mathcal{CCLLS}(u)$ are proved in a similar way. □

7 Construction of the tree of shapes by fusion of upper and lower trees

Because of Lemma 5.2, we may write the statement (ix) of Lemma 2.8 when A is an upper or lower level set of a weakly oscillating function.

Lemma 7.1. *Let $u \in C(\overline{\Omega})$ be a weakly oscillating function. If $X \in \mathcal{CC}([u \geq \lambda])$ and Y is an internal hole of X, then there is $O \in \mathcal{CC}([u < \lambda])$ such that $Y = \mathrm{Sat}(O)$. Similarly, if $X \in \mathcal{CC}([u < \lambda])$ and Y is an internal hole of X, then there is $O \in \mathcal{CC}([u \geq \lambda])$ such that $Y = \mathrm{Sat}(O)$.*

Since the shapes of the tree $\mathcal{S}(u)$ are the saturations of the nodes of $\mathcal{CCULS}(u)$ and $\mathcal{CCLLS}(u)$, we can construct $\mathcal{S}(u)$ by fusing the information of the upper and lower trees. This operation can de done very simply because of the precise branch structure of both trees described in Propositions 5.7 and 5.8. The overall procedure is illustrated in Figure 6.

Since we are going to use simultaneously intervals of both trees, we shall denote by $[A, B]_{ut}$ the interval of the upper tree determined by nodes $A, B \in \mathcal{CCULS}(u)$, and $[A, B]_{lt}$ the interval of the lower tree determined by nodes $A, B \in \mathcal{CCLLS}(u)$.

To fix ideas, let us assume that p_∞ is the global minimum of u and let $\Lambda = \mathrm{cc}([u = \inf_{x\in\overline{\Omega}} u(x)], p_\infty)$. Observe that Λ is a leaf of $\mathcal{CCLLS}(u)$. Let $\mathcal{CCLLS}_\Lambda(u) = \mathcal{CCLLS}(u) \setminus [\Lambda, \overline{\Omega}]_{lt}$. All nodes of $[\Lambda, \overline{\Omega}]_{lt}$ have $\overline{\Omega}$ as saturation. If $C \in \mathcal{CCLLS}_\Lambda(u)$, then all nodes previous to it do not contain Λ. Thus, $\mathcal{CCLLS}_\Lambda(u)$ is a union of maximal branches of $\mathcal{CCLLS}(u)$. Notice that the only node of $\mathcal{CCULS}(u)$ containing p_∞ is $\overline{\Omega}$.

The tree $\mathcal{CCULS}(u)$ and $\mathcal{CCLLS}_\Lambda(u)$ are broken into maximal branches of the form $[A, B]_{it}$, $i = l, u$, where A is either a leaf or a bifurcating node and B may be a bifurcating node or not, in which case it coincides with $\overline{\Omega}$. As we have pointed out in Proposition 6.5, the maximal branches start and end at the singularities of the trees. Each branch $\mathcal{B} = [A, B]_{it}$, $i \in \{u, l\}$, determines the set of shapes $\mathrm{Sat}(\mathcal{B}) := \{\mathrm{Sat}(C) : C \in \mathcal{B}\}$ of $\mathcal{S}(u)$.

Given a maximal branch $\mathcal{B}^+ = [A, B]_{ut}$ of $\mathcal{CCULS}(u)$, let us join to it the corresponding branches of $\mathcal{CCLLS}(u)$. First assume that A is a leaf (at level λ) of $\mathcal{CCULS}(u)$. If A has no internal holes, then $A = \mathrm{Sat}(A)$ is a shape of upper type. If A has internal holes $Y_1, \ldots, Y_r$ (they do not contain p_∞), by Lemma 7.1, they correspond to saturations of connected components of lower level sets of u, i.e., to saturations of nodes in $\mathcal{CCLLS}(u)$. Let $O_1, \ldots, O_r \in \mathcal{CC}([u < \lambda])$ be such that $Y_j = \mathrm{Sat}(O_j)$, $j = 1, \ldots, r$. Since by Proposition 5.5, A is a regional maximum of u with internal holes, then each O_j is a bifurcating node of $\mathcal{CCLLS}(u)$. Moreover, since O_j contains a regional minimum it is the terminal node of a maximal branch $\mathcal{B}_j^-$ of $\mathcal{CCLLS}_\Lambda(u)$. We attach to $\mathrm{Sat}(A)$ the set of shapes of lower type $\mathrm{Sat}(\mathcal{B}_j^-)$. Finally, if A is a bifurcating node, then we may treat A as we treat B'; let us explain this. Recall that B is a bifurcating limit node with bifurcating node $B' = \mathrm{cc}([u \geq \lambda], B)$. Let $\mathcal{B}_k^+ = [A_k, B_k]_{ut}$, $k = 1, \ldots, s$, be all maximal branches of $\mathcal{CCULS}(u)$ which bifurcate at B' (one of them coincides with $\mathcal{B}^+$). Notice that $B_i' = \mathrm{cc}([u \geq \lambda], B_i) = B'$ for any $i \in \{1, \ldots, s\}$. Let $Z_i = \mathrm{Sat}(U_i)$ where $U_i \in \mathcal{CC}([u < \lambda])$, $i = 1, \ldots, q$, be all internal holes of negative type of B'. Notice that U_i is not necessarily the terminal node of a maximal branch $\mathcal{B}_i^-$ of $\mathcal{CCLLS}_\Lambda(u)$.

Lets us link the families $\mathrm{Sat}(\mathcal{B}_k^+)$ and $\mathrm{Sat}(\mathcal{B}_i^-)$ to the tree of shapes. By reordering if necessary, we may assume that $\{\mathrm{Sat}(B_k) : k = 1, \ldots, p\}$ are disjoint and $\{\mathrm{Sat}(B_j) : j = p+1, \ldots, s\}$ are contained in some $\mathrm{Sat}(B_k)$ for $k \in \{1, \ldots, p\}$. The families of shapes $\mathrm{Sat}(\mathcal{B}_k^+)$, $k = 1, \ldots, p$, are linked to $\mathrm{Sat}(B')$. For each $j \in \{p+1, \ldots, s\}$, $B_j \in \mathcal{CC}([u > \lambda])$ and determines a hole of a connected component of $[u \leq \lambda]$, thus it is a hole of a limit node of $\mathcal{CCLLS}(u)$.

Observe that we have $\mathrm{Sat}(B_k) \neq \overline{\Omega}$ for all $k = 1, \ldots, s$. Let $j \in \{p+1, \ldots, s\}$ and let $k_j \in \{1, \ldots, p\}$ be such that $\mathrm{Sat}(B_j) \subseteq \mathrm{Sat}(B_{k_j})$. Then $\mathrm{Sat}(B_j)$ is a hole in a limit node $N \in \mathcal{CC}([u \leq \lambda])$ of $\mathcal{CCLLS}(u)$. Moreover $N \subseteq \mathrm{Sat}(B_{k_j})$, and therefore N is a limit node

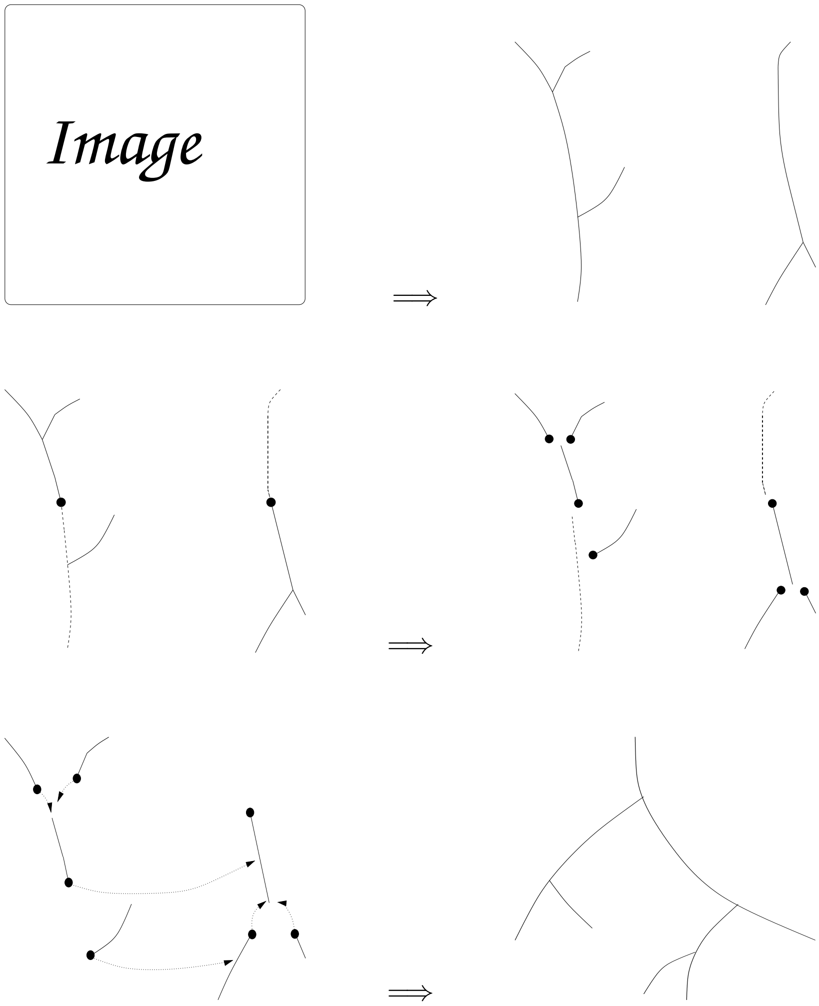

Figure 6: Top: a) Left: A given image. b) Right: the upper and lower level set trees of the image. Medium: c) Left: We identify the set of shapes that contain p_∞ (dotted line) and d) Right: the maximal branches that have to be linked with the other tree (black dots) in both trees. Bottom: e) Left: We link the maximal branches of both trees. f) Right: The final tree of shapes.

of $\mathcal{CCLLS}_\Lambda(u)$. Then, we attach $\mathrm{Sat}(\mathcal{B}_j^+)$ to the limit shape $\mathrm{Sat}(N)$. The previous argument applies in both cases: $\lambda = \inf u$, or $\lambda > \inf u$.

Even if the shapes corresponding the negative internal holes of B' will be considered with the lower tree, let us make some comments. If the internal hole Z_i is disjoint to all $\mathrm{Sat}(B_k)$, $k \in \{1, \ldots, p\}$, then U_i is the terminal node of a maximal branch $\mathcal{B}_i^-$ of $\mathcal{CCLLS}_\Lambda(u)$. We link $\mathrm{Sat}(\mathcal{B}_i^-)$ to $\mathrm{Sat}(B')$. If $Z_i = \mathrm{Sat}(U_i)$ is contained in some $\mathrm{Sat}(B_k)$, $k \in \{1, \ldots, p\}$, then U_i is present in $\mathcal{CCLLS}_\Lambda(u)$ and will be handled with the lower tree in next paragraph.

Given a maximal branch $\mathcal{B}^- = [A, B]_{lt}$ of $\mathcal{CCLLS}_\Lambda(u)$, let us link it with the corresponding branches of $\mathcal{CCULS}(u)$. First assume that A is a leaf (at level λ) of $\mathcal{CCLLS}(u)$. If A has no internal holes, then $A = \mathrm{Sat}(A)$ is a limit shape of lower type. If A has internal holes, they are of positive type, i.e., they correspond to saturations of connected components $B_1, \ldots, B_r$ of $[u > \lambda]$. The sets B_j are bifurcation limit nodes which are terminal in maximal branches $\mathcal{B}_j^+$ of of the tree $\mathcal{CCULS}(u)$. Observe that any two $\mathrm{Sat}(B_j)$ cannot be nested since they are internal holes of A. As we explained for the upper tree, we attach $\mathrm{Sat}(\mathcal{B}_j^+)$ to $\mathrm{Sat}(A)$.

Now, let us consider the case where A is a bifurcating node (not a leaf). In this case $A \in \mathcal{CC}([u \leq \lambda])$ and there are at least two connected components of $[u < \lambda]$ in A. Observe that, since A is an initial (limit) node of a maximal branch of $\mathcal{CCLLS}_\Lambda(u)$, then all branches arriving to it are also in $\mathcal{CCLLS}_\Lambda(u)$, as is $\mathcal{B}^-$. Thus, we repeat the arguments that we used for the bifurcating limit nodes B' of $\mathcal{CCULS}(u)$: we link the maximal branches of $\mathcal{CCLLS}_\Lambda(u)$ which arrive at $\mathrm{Sat}(A)$. Notice that the positive holes have already been linked when we considered $\mathcal{CCULS}(u)$.

Finally, let us consider the bifurcating node $B \in \mathcal{CC}([u < \lambda])$ at level $\lambda \in I\!R$ and its associated bifurcating limit node $B' = \mathrm{cc}([u \leq \lambda], B)$. If $B' \in \mathcal{CCLLS}_\Lambda(u)$ we proceed as in the previous case. Thus, we are lead to assume that $B' \supseteq \Lambda$. Let us consider the maximal branches $\mathcal{B}_k^-$ in $\mathcal{CCLLS}_\Lambda(u)$ ending at B_k (one of them ends in B) which arrive at B', $k = 1, \ldots, r$, for some r. Let $\mathcal{B}_0^-$ the maximal branch of $\mathcal{CCLLS}(u)$ ending at $B_0 \in \mathcal{CC}([u < \lambda])$ where $B_0 \supseteq \Lambda$ and $B_0 \subseteq B'$. Each B_k, $k = 1, \ldots, r$, determines a negative hole of a connected component C_k of $[u \geq \lambda]$. We link $\mathrm{Sat}(\mathcal{B}_k^-)$ to $\mathrm{Sat}(C_k)$. Observe that any connected component of $[u \leq \lambda]$ which contains any of the B_k's contains also B_0 and its saturation is the whole $\overline{\Omega}$. Thus we cannot link any of the families $\mathrm{Sat}(\mathcal{B}_k^-)$ to a lower shape and we have to link them to a upper shape. Notice that this argument also covers the situation in which a set $\mathrm{Sat}(B_i) \subseteq \mathrm{Sat}(B_j)$, $i, j \in \{1, \ldots, r\}$, $i \neq j$.

Observe that: 1) the final structure contains the saturations of all connected components of upper and lower level sets, 2) there are no cycles since if we have linked $\mathrm{Sat}(A)$ to $\mathrm{Sat}(B)$ is because $\mathrm{Sat}(A) \subseteq \mathrm{Sat}(B)$, 3) If $A \in \mathcal{CCLLS}(u)$, $B \in \mathcal{CCULS}(u)$, or viceversa, and we have linked $\mathrm{Sat}(A)$ to $\mathrm{Sat}(B)$, we have chosen $\mathrm{Sat}(B)$ to be the minimal shape containing A.

8 A quick overview of the literature

The use of a topographic description of images, surfaces, or $3D$ data has been introduced and motivated in different areas of research, including image processing, computer graphics, and geographic information systems (GIS), e.g., [2, 4, 1, 6, 9, 11, 14, 17, 20, 24, 22, 26, 27, 30]. The motivations for such a description differ depending on the field of application. In all cases these descriptions aim to achieve an efficient description of the basic shapes in the given image and their topological changes as a function of a physical quantity that depends on the type of data (height in data elevation models, intensity in images, etc.).

In computer graphics and geographic information systems, topographic maps represent a high level description of the data. Topographic maps are represented by the contour maps, i.e., the isocontours of the given scalar data. The contour map is organized in a data structure, either the *contour tree* [17, 15], or the *Reeb graph* [33, 25]. The contour tree represents the nesting of contour lines of the contour map. According to [17], each node represents a connected component of an upper (or lower) level set $[u \geq \lambda]$ ($[u \leq \lambda]$), and links between nodes represent a parent-child relationship, a link going from the containing to the contained set in the upper tree, or viceversa if we consider the lower tree. The contour tree can be considered as an implementation of Morse theory, in the sense that it encodes the topological changes of the isocontours of the data, and relates these topological changes to the criticalities of the function. For practical applications, the data structure has to be implemented with a fast algorithm and with minimal storage requirements. In [15] this is accomplished with a variant of the contour tree where the criticalities (maxima, minima, saddles, computed in a local way) are computed first.

A related data structure is the *Reeb graph*, which represents the splitting and merging of the isocontours. The *Reeb graph* of the height function u is obtained by identifying two points $p, q \in \Omega$ such that $u(p) = u(q)$ if they are in the same connected component of the isocontour $[u = u(p)]$. Thus, a cross-sectional contour corresponds to a point of an edge of the Reeb graph, and a vertex represents a critical point of the height function u. The Reeb graph was proposed in [33] as a data structure for encoding topographic maps. In the context of computer graphics, Morse theory has also been used to encode surfaces in $3D$ space [30]. In [30], the authors also use a tree structure like the Reeb graph complemented with information about the Morse indexes of the singularities and including enough (information about) intermediate contours to be able to reconstruct by interpolation the precise way in which the surface is embedded in $3D$ space.

In image processing, the topographic description was advocated as a local and contrast invariant description of images (i.e., invariant under illumination changes), and has lead to an underlying notion of shapes of an image as the family of connected components of upper or lower level sets of the image [6, 27]. An efficient description of the family of shapes in terms of a tree was proposed in [24, 22], further developed in as a tree of level lines (or isocontours of the interpolated image) [19] and studied in [2, 22]. A different but equivalent approach (for 2D images) was presented in [32]. The work in [16] can be considered as a mathematical description of the (iso) contour tree in the case of two-dimensional functions.

In [11], Morse theory has also been used as a basic model to describe the geometric structures of $2D$ and $3D$ images, and in general, of multidimensional data, assuming that the given data are interpolated by a continuous real valued functions. Applications have been given in different domains, in particular, to visualize structures in $3D$ medical images.

Acknowledgement. The first and second authors acknowledge partial support by PNPGC project, reference MTM2006-14836. We would like to thank A. Frangi and E. Zacur for many useful discussions during the preparation of this paper.

References

[1] C. Bajaj, V. Pascucci, and D.R. Schikore. Fast isocontouring for improved interactivity. *In Proc. IEEE Symp. on Volume Visualization, San Francisco, Oct. 7-8*, pages 39–46, 1996.

[2] C. Ballester, V. Caselles, and P. Monasse, "The tree of shapes of an image". *ESAIM: Control, Opt. and Calc. of Variations* **9**, pp. 1–18, 2003.

[3] C. Ballester, V. Caselles, L. Igual, and L. Garrido "Level Lines Selection with Variational Models for Segmentation and Encoding'". To appear in Journal Math. Imaging and Vision.

[4] P.J. Besl and R.C. Jain. Segmentation through symbolic surface descriptions. *CVPR, May*, 1986.

[5] F. Cao, Y. Gousseau, J.M. Morel, P. Musé, and F. Sur, "An a contrario decision method for shape element recognition". Preprint CMLA No. 16, 2004.

[6] V. Caselles, B. Coll, and J.M. Morel, "Topographic maps and local contrast changes in natural images". *Int. J. Comp. Vision* **33**, pp. 5–27, 1999.

[7] V. Caselles, and P. Monasse. Grain Filters. *Journal of Mathematical Imaging and Vision*, 17, pp. 249-270, 2002.

[8] V. Caselles, and P. Monasse, "Geometric Description of Topographic Maps and Applications to Image Processing". *In preparation.*

[9] V. Caselles, G. Sapiro, A. Solé, and C. Ballester, "Morse Description and Morphological Encoding of Continuous Data". *SIAM J. on Multiscale Model. and Simulation* **2**, pp. 179–209, 2004.

[10] R. Chiariglioni, "MPEG and Multimedia Communitations". *IEEE Transactions on Circuits and Systems for Video Technology* **7**, pp. 5-18, 1997.

[11] J.L. Cox and D.B. Karron. Digital morse theory with suggested applications. *Manuscript available at http://www.casi.net*, 1998.

[12] A. Desolneux, L. Moisan, and J.M. Morel. Edge detection by helmholtz principle. *Journal of Mathematical Imaging and Vision*, 14(3):271–284, 2001.

[13] F. Guichard and J.M. Morel, *Image Iterative Smoothing and P.D.E.'s'.* Book in preparation, 2000.

[14] R. Haralick, L. Winston, and T. Laffey. The topographic primal sketch. *Int. J. Rob. Research*, 2, 1983.

[15] M. van Kreveld, R. va Oostrum, V. Bajaj, V. Pascucci, and D. Schikore. Contour trees and small seed sets for isosurface traversal. *In 13th ACM Symposium on Computational Geometry*, pages 212–220, 1997.

[16] A.S. Kronrod. On functions of two variables. *Uspehi Mathematical Sciences (NS)*, 35(5):24–134, 1950.

[17] I.S. Kweon and T. Kanade. Extracting topograpic terrain features from elevation maps. *CVGIP: Image Understanding*, 59(2):171–182, 1994.

[18] C. Kuratowski. *Topologie I, II.* Editions J. Gabay, Paris, 1992.

[19] J.L. Lisani. Comparaison automatique d'images par leurs formes. *Ph.D Thesis, Université de Paris-Dauphine, July*, 2001.

[20] J.L. Lisani, L. Moisan, P. Monasse, and J.M. Morel. Affine invariant mathematical morphology applied to a generic shape recognition algorithm. *Comp. Imaging and Vision*, 18, 2000.

[21] E. Meinhardt, V. Caselles, E. Zacur, and A. Frangi. *The tree of shapes of a 3D image. Applications to medical image segmentation.* In preparation.

[22] P. Monasse. *Représentation morphologique d'images numériques et application au recalage.* PhD thesis, Université de Paris-Dauphine, 2000.

[23] P. Monasse, "Contrast invariant image registration". In: *Proc. of International Conference on Acoustics, Speech and Signal Processing* **6**, pp. 3221–3224, 1999.

[24] P. Monasse and F. Guichard, "Fast computation of a contrast invariant image representation". *IEEE Transactions on Image Processing* **9**, pp. 860–872, 2000.

[25] G. Reeb. Sur les poits singuliers d'une forme de pfaff completement integrable ou d'une fonction numérique. *Comptes Rendus Acad. Sciences Paris*, 222:847–849, 1946.

[26] P. Salembier and L. Garrido, "Binary partition tree as an efficient representation for image processing, segmentation, and information retrieval". *IEEE Transactions on Image Processing* **9**(4), pp. 561–576, 2000.

[27] P. Salembier and J. Serra, "Flat zones filtering, connected operators and filters by reconstruction". *IEEE Transactions Image Processing* **4**(8), pp. 1153-1160, 1995.

[28] H.H. Schaefer, *Banach lattices and positive operators*, Springer Verlag, 1974.

[29] J. Serra, *Image analysis and mathematical morphology.* Academic Press: New York, 1982.

[30] Y. Shinagawa, T.L. Kunii, and Y.L. Kergosien. Surface coding based on morse theory. *IEEE Computer Graphics and Appl.*, 11(5):66–78, 1991.

[31] T. Sikora, "The MPEG-7 visual standard for content description - an overview". *IEEE Transactions on Circuits and Systems for Video Technology* **11**, pp. 696-702, 2001.

[32] Y. Song and A. Zhang, "Monotonic Trees and its Application to Image Processing". *10th International Conf. on Discrete Geometry for Computer Imagery*, Bordeaux, France, 2002.

[33] S. Takahashi, T. Ikeda, Y. Shinagawa, T. Kunii, and M. Ueda. Algorithms for extracting correct critical points and constructing topological graphs from discrete geographic elevation data. *Eurographics 95*, 14(3):181–192, 1995.

[34] M. Wertheimer, "Untersuchungen zur Lehre der Gestalt, II'. *Psychologische Forschung* **4**, pp. 301–350, 1923.