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Anisotropic Cheeger Sets and Applications*

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Abstract. The main purpose of this paper is to develop the mathematical analysis of anisotropic total variation problems with a degenerate metric and the computation of the associated Cheeger sets. We illustrate our analysis with the computation of Cheeger sets with respect to different anisotropic norms of relevance in applications to image processing. In particular, we describe the computation of global minima of geodesic active contour models, and we illustrate the use of Cheeger sets for the problem of edge linking.

Key words. Cheeger sets, anisotropic total variation, active contours, edge linking

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1. Introduction.

Given a nonempty open bounded subset Ω of $\mathbb{R}^N$, we call Cheeger constant of Ω the quantity

$$\mathcal{C}_\Omega := \min_{F\subseteq\Omega} \frac{P(F)}{|F|}. \tag{1.1}$$

Here $|F|$ denotes the N-dimensional volume of F, and $P(F)$ denotes the perimeter of F. The minimum in (1.1) is taken over all nonempty sets of finite perimeter contained in Ω. A Cheeger set of Ω is any set $G \subseteq \Omega$ which minimizes (1.1). Observe that G is a Cheeger set of Ω if and only if $|G| > 0$ and G minimizes

$$\min_{F\subseteq\Omega} P(F) - \mathcal{C}_\Omega |F|. \tag{1.2}$$

Existence of Cheeger sets follows directly from the direct methods of calculus of variations. Uniqueness of Cheeger sets is a more delicate issue and is not true in general (a counterexample is given in [39] when Ω is not convex), though it has recently been proved that it is generically true [19] (that is, true modulo a small perturbation of the domain Ω). However, uniqueness of Cheeger sets inside convex bodies of $\mathbb{R}^N$ was proved in [20] when the convex body is uniformly convex and of class C^2 and in [2] in the general case. The case of convex bodies of $\mathbb{R}^2$ was studied in [3, 39].

The computation of Cheeger sets has recently been the object of several papers [15, 17]. One of the possible algorithms consists in solving the variational problem

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$$(1.3)\qquad \min_{u\in BV(\Omega)\cap L^2(\Omega)} \left\{ \int_\Omega |Du| + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1} + \frac{\lambda}{2}\int_\Omega (u-1)^2\, dx \right\}.$$

This strictly convex lower semicontinuous functional has a unique minimizer $u \in BV(\Omega) \cap L^2(\Omega)$ satisfying $0 \le u \le 1$. Moreover, for any $s \in (0,1]$ the level set $E_s := \{x \in \Omega : u(x) \ge s\}$ is a solution of

$$(1.4)\qquad \min_{F\subseteq\Omega} P(F) - \mu|F|,$$

where $\mu := \lambda(1-s)$ and the infimum is taken over the sets $F \subseteq \Omega$ of finite perimeter in $\mathbb{R}^N$ [4, 21]. When taking $\lambda \in (0,+\infty)$ and $s \in (0,1]$ we are able to cover the whole range of $\mu \in [0,\infty)$ [4]. Since the family of level sets E_s is nested, the solution of (1.4) is unique for any $\mu \in (0,+\infty)$ up to a countable exceptional set. Moreover, when λ is big enough, the level set associated to the maximum of u, $\{x \in \Omega : u(x) = \|u\|_\infty\}$, is the maximal Cheeger set of Ω [4, 20]. Observe that this provides an algorithm for computing the maximal Cheeger set (and also the solution of the family of problems (1.4)). In particular, using Chambolle's algorithm [26] to minimize (1.4), one passes to a dual variational problem which can be solved by a simple iterative scheme.

Our purpose in this paper is to study Cheeger sets in the context of image processing, in particular, their connections with active contours and edge linking. For that we use the theory of anisotropic perimeters developed in [5, 10] to extend model (1.3) to general anisotropic perimeters, including as particular cases the geodesic active contour model with an inflating force [22, 23, 41], a model for anisotropic diffusion, and a model for edge linking. Thus, we will use the results in [5, 10] to study the problem

$$(1.5)\qquad \min_u \int_\Omega |Du|_\phi + \int_{\partial\Omega} \phi(x,\nu^\Omega)|u|\, d\mathcal{H}^{N-1} + \frac{\lambda}{2}\int_\Omega (u-1)^2\, h\, dx,$$

where $\phi : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a metric integrand in Ω which is symmetric in ξ, i.e., $\phi(x,-\xi) = \phi(x,\xi)$ for any $\xi \in \mathbb{R}^N$ and any $x \in \Omega$; $\int_\Omega |Du|_\phi$ denotes the anisotropic total variation with respect to the integrand ϕ (see section 3); and $h \in L^\infty(\Omega)$, $h(x) > 0$ a.e., with $\int_\Omega \frac{1}{h(x)}\, dx < \infty$. We denote by ν^Ω the outer unit normal to Ω at points of $\partial\Omega$. To shorten the expressions inside the integrals we shall write h, u instead of $h(x), u(x)$, with the only exception being $\phi(x,\nu^\Omega)$. Again we prove that if u is a solution of (1.5), then the level sets $\{x \in \Omega : u(x) \ge s\}$, $s \in (0,1]$, are solutions of

$$(1.6)\qquad \min_{F\subseteq\Omega} P_\phi(F) - \mu|F|_h,$$

where $\mu := \lambda(1-s)$, $P_\phi(F)$ is the anisotropic perimeter of F (see section 3), and $|F|_h = \int_F h(x)\, dx$. As in the Euclidean case, the solution of (1.6) is unique for any $s \in (0,1]$ up to a countable exceptional set. Moreover, when λ is big enough, the level set associated to the maximum of u, $\{x \in \Omega : u(x) = \|u\|_\infty\}$, is the maximal (ϕ,h)-Cheeger set of Ω. A (ϕ,h)-Cheeger set in Ω is a minimizer of the problem

$$(1.7)\qquad \inf\left\{ \frac{P_\phi(F)}{|F|_h} : F \subseteq \overline{\Omega} \text{ of finite perimeter, } |F|_h > 0 \right\},$$

and the value of this infimum is the (ϕ, h)-Cheeger constant, denoted by $\mathcal{C}_\Omega^{\phi,h}$. The computation of the maximal (ϕ, h)-Cheeger set (together with the solution of the family of problems (1.6)) can be computed using Chambolle's algorithm [26]. To simplify our expressions, instead of (ϕ, h)-Cheeger set and constant, we will say ϕ-Cheeger set and constant.

In order to develop the previous approach we have to assume that the metric integrand $\phi(x,\xi)$ is continuous and coercive near the boundary of Ω, which amounts to saying that for $x \in \Omega$ near $\partial\Omega$ we have

$$\alpha|\xi| \le \phi(x,\xi) \le \beta|\xi| \qquad \forall \xi \in \mathbb{R}^N,\ 0 < \alpha < \beta. \tag{1.8}$$

Then we adapt the results in [5, 10] in order to study the variational problem (1.5). This problem was studied in [49] for the case when ϕ is continuous and satisfies (1.8) for all $(x,\xi) \in \Omega \times \mathbb{R}^N$. In case that ϕ is coercive everywhere we can minimize (1.5) in the space of functions of bounded variation $BV(\Omega)$. In the present situation, where we do not assume that ϕ is coercive, we have to extend this space and consider functions $u \in L^1(\Omega)$ such that $\int_\Omega |Du|_\phi < \infty$. We denote this space by $BV_\phi(\Omega)$. Observe that, since we are studying the Dirichlet problem, we need that functions $u \in BV_\phi(\Omega)$ have a trace on $\partial\Omega$. Thus, by assuming that ϕ is coercive near the boundary of Ω, the finiteness of the anisotropic total variation implies that u is a bounded variation function near $\partial\Omega$ and therefore has a trace on $\partial\Omega$. The interest in degenerate (noncoercive) metric integrands ϕ comes from the applications, where it can be natural to assume that ϕ vanishes on a subset of Ω (arcs of curve if $N = 2$ or surface patches if $N = 3$).

We illustrate this formalism with three examples: (a) the geodesic active contour model; (b) the anisotropic diffusion case; and (c) a model for edge linking.

(a) *The geodesic active contour model.* Let $I : \Omega \to \mathbb{R}^+$ be a given image in $L^\infty(\Omega)$, G be a Gaussian function, and

$$g(x) = \frac{1}{\sqrt{1 + |\nabla(G * I)|^2}} \tag{1.9}$$

(where in $G * I$ we have extended I to $\mathbb{R}^N$ by taking the value 0 outside Ω). Observe that $g \in C(\overline{\Omega})$ and $\inf_{x\in\overline{\Omega}} g(x) > 0$. The geodesic active contour model [22, 23, 41] with an inflating force (see [30]) corresponds to the case where $\phi(x,\xi) = g(x)|\xi|$, $|Du|_\phi = g(x)|Du|$, and $h(x) = 1$, $x \in \Omega$. The purpose of this model is to locate the boundary of an object of the image at the points where the gradient is large. The presence of the inflating term helps to avoid minima collapsing into a point. The analysis of this model was done in [18, 24, 41] using the level set formulation of (1.6). In this case we write $P_g(F)$ instead of $P_\phi(F)$, and we have $P_g(F) := \int_{\partial^* F} g \, d\mathcal{H}^{N-1}$, where $\partial^* F$ is the reduced boundary of F [7].

In this case the ϕ-Cheeger sets are a particular instance of geodesic active contours with a constant inflating force $\mu = \mathcal{C}_\Omega^{g,1}$. An interesting feature of this formalism is that it permits us to define local ϕ-Cheeger sets as local (regional) maxima of the function u. They are ϕ-Cheeger sets in a subdomain of Ω. They can be identified with boundaries of the image, and the above formalism permits us to compute several active contours at the same time (the same holds true for the edge linking model).

A more general active contour model, based on Finsler metrics, was introduced in [48]. In this paper, the authors minimized the Finsler metric using dynamic programming. A different numerical approach based on graph cuts, and valid for submodular Finsler metrics, was proposed in [42].

(b) *An anisotropic diffusion model.* The model (1.5) contains the case $\phi(x,\xi) = |A_x\xi|$, where A_x is a symmetric positive definite matrix for each $x \in \Omega$. A particular instance when $N = 2$ is the anisotropic diffusion model given by $A_x = V^\perp(x) \otimes V^\perp(x) + g(x)V(x) \otimes V(x)$, where $V(x) = \frac{\nabla I(x)}{\sqrt{1+|\nabla I(x)|^2}}$ and $V(x)^\perp$ denotes the counterclockwise rotation of $V(x)$ of angle $\frac{\pi}{2}$. Notice that by the structure of A_x we could also take the clockwise rotation.

(c) *An edge linking model.* Another interesting application of the above formalism is to edge linking. Given a set $\Gamma \subseteq \Omega$ (which may be the output of an edge detector formed by arcs of curve if $\Omega \subseteq \mathbb{R}^2$ or surface patches if $\Omega \subseteq \mathbb{R}^3$), we define $d_\Gamma(x) = \mathrm{dist}(x,\Gamma)$ and the metric integrand $\phi(x,\xi) = d_\Gamma(x)|\xi|$. In this case, we experimentally see that the ϕ-Cheeger set determined by this weighted metric has a boundary formed by a set of curves ($N = 2$) or surfaces ($N = 3$) linking Γ.

Let us mention the formulation of active contour models without edges proposed in [28] by Chan and Vese, whose solution can be related to the general formulation (1.6). Let $I : \Omega \to \mathbb{R}^+$ be a given image and $g \in C(\overline{\Omega})$ be such that $\inf_{x\in\overline{\Omega}} g(x) > 0$. The authors proposed minimizing

$$(1.10) \qquad \min_{F\subseteq\Omega, c_1,c_2\in\mathbb{R}} \mathcal{E}_g(F, c_1, c_2) := P_g(F;\Omega) + \lambda \int_F (I(x) - c_1)^2\, dx + \lambda \int_{\Omega\setminus F} (I(x) - c_2)^2\, dx,$$

where the minimum is taken over the sets F of finite perimeter in Ω, with $c_1, c_2 \in \mathbb{R}$, $\lambda > 0$, and $P_g(F,\Omega) := \int_{\partial^* F\cap\Omega} g\, d\mathcal{H}^{N-1}$ is the weighted perimeter of F in Ω. Although in the initial proposal [28] the authors took $g = 1$ (in this case (1.10) is the restriction of the Mumford–Shah functional to a binary segmentation of the image), the extension to a weighted perimeter was natural and has been considered, for instance, in [13]. If the set F is fixed, then the minimum of $\mathcal{E}_g(F, c_1, c_2)$ with respect to $c_1, c_2 \in \mathbb{R}$ gives us the values $\overline{c}_1 = \frac{\int_F I(x)\,dx}{|F|}$ and $\overline{c}_2 = \frac{\int_{\Omega\setminus F} I(x)\,dx}{|\Omega\setminus F|}$, and we may write

$$(1.11) \qquad \min_{F\subseteq\Omega, c_1,c_2\in\mathbb{R}} \mathcal{E}_g(F, c_1, c_2) = \min_{F\subseteq\Omega} \mathcal{E}_g(F, \overline{c}_1, \overline{c}_2).$$

Observe that

$$\mathcal{E}_g(F, \overline{c}_1, \overline{c}_2) = P_g(F,\Omega) + \lambda \int_F ((I(x) - c_1)^2 - (I(x) - c_2)^2)\, dx + \lambda \int_\Omega (I(x) - c_2)^2\, dx$$

and that it suffices to minimize with respect to F the first two terms of this sum, the last one being a constant. Now, as proposed in [27, 13], for any fixed values $c_1, c_2 \in \mathbb{R}$ the global minimizer of $\mathcal{E}_g(F, \overline{c}_1, \overline{c}_2)$ with respect to F can be found by solving the convex optimization problem [27, Theorem 2]

$$(1.12) \qquad \min_{0\le u\le 1} \int_\Omega g|Du| + \lambda \int_\Omega ((I(x) - c_1)^2 - (I(x) - c_2)^2)\, u\, dx$$

and then setting $F := \{u > t\}$ for a.e. $t \in (0, 1)$. Thus, iterating between the solution of (1.12) with fixed values of c_1, c_2 and the updating of c_1, c_2 just described, we have a two step algorithm to solve the Chan–Vese model [27, 13].

Let us also mention the interesting work [51] in which the authors propose a method for convexifying the total variation regularization of some nonlinear and nonconvex data attachment terms (e.g. the computation of disparity in rectified stereo pairs) which leads to the solution of an anisotropic total variation problem with Dirichlet-type boundary conditions. The present paper provides the mathematical foundations of that approach in the case of degenerate anisotropies.

For more information on Cheeger sets and their role in estimating the first eigenvalue of the p-Laplacian for $p \in [1, \infty)$, we refer the reader to [29, 43, 37, 38]. These results have been extended in several directions, in particular, using weighted volume and perimeter [16, 15] and for anisotropic versions of the perimeter [40].

Let us describe the plan of this paper. In section 2 we collect some preliminaries about functions of bounded variation and Green's formula. In section 3 we recall the definition of anisotropic total variation with respect to a metric integrand and adapt the lower semicontinuity and relaxation results in [5, 10] required to study the variational problem (1.5). In section 4 we extend Green's formula for bounded vector fields $\sigma : \Omega \to \mathbb{R}^N$ with divergence in $L^\infty(\Omega)$ and functions $u \in BV_\phi(\Omega)$. Though these results are classical [9] in the usual $BV(\Omega)$ case and the corresponding extensions to BV_ϕ functions only require the right definitions, for the reader's convenience and for future reference we include in Appendix A a sketch of the proof of the main results. In section 6 we study the existence and uniqueness of solutions of problem (1.5) and prove that the level sets of the solution u solve problem (1.6). Then we prove that the level set $\{x \in \Omega : u(x) = \|u\|_\infty\}$ is the maximal ϕ-Cheeger set of Ω. In section 7 we briefly sketch the existence and uniqueness of solutions of the problem analogous to (1.5), replacing Dirichlet with Neumann boundary conditions. In section 8 we extend Chambolle's algorithm to cover the case of certain degenerate (or not) anisotropic norms. In section 9 we describe the computation of ϕ-Cheeger sets in floating-point images and we describe the numerical computation of the ϕ-perimeter of the level set of a digital image. In section 10 we display experiments to illustrate the computation of global minima of the geodesic active contour model with an inflating force and of the edge linking model. In all these cases we display the corresponding ϕ-Cheeger sets. In section 11 we apply the framework developed in the previous sections to a model of anisotropic diffusion along the level lines of a given image. Finally, section 12 summarizes our main conclusions. Appendix A contains the proof of several results that have not been included in the main body of the text.

2. Preliminaries.

2.1. Bounded variation functions and sets of finite perimeter.

Let Ω be an open subset of $\mathbb{R}^N$. A function $u \in L^1(\Omega)$ whose gradient Du in the sense of distributions is a (vector-valued) Radon measure with finite total variation in Ω is called a function of bounded variation. The class of such functions will be denoted by $BV(\Omega)$. The total variation of Du on Ω turns out to be

$$\sup\left\{\int_\Omega u \,\operatorname{div} z\, dx : z \in C_0^\infty(\Omega;\mathbb{R}^N), \|z\|_{L^\infty(\Omega;\mathbb{R}^N)} := \operatorname*{ess\,sup}_{x\in\Omega} |z(x)| \le 1\right\} \tag{2.1}$$

(where for a vector $v = (v_1, \ldots, v_N) \in \mathbb{R}^N$ we set $|v|^2 := \sum_{i=1}^N v_i^2$) and will be denoted by $|Du|(\Omega)$ or by $\int_\Omega |Du|$. It turns out that the map $u \to |Du|(\Omega)$ is $L^1_{\text{loc}}(\Omega)$-lower semicontinuous. The total variation of u on a Borel set $B \subseteq \Omega$ is defined as $\inf\{|Du|(A) : A \text{ open}, B \subseteq A \subseteq \Omega\}$. For more results and information on functions of bounded variation we refer the reader to [7, 32].

A measurable set $E \subseteq \mathbb{R}^N$ is said to be of finite perimeter in Ω if (2.1) is finite when u is replaced by the characteristic function χ_E of E. The perimeter of E in Ω is defined as $P(E, \Omega) := |D\chi_E|(\Omega)$, and $P(E, \Omega) = P(\mathbb{R}^N \setminus E, \Omega)$. We shall use the notation $P(E) := P(E, \mathbb{R}^N)$. For sets of finite perimeter E one can define the essential boundary $\partial^* E$, which is countably $(N-1)$-rectifiable with finite $\mathcal{H}^{N-1}$ measure, and compute the outer unit normal $\nu^E(x)$ at $\mathcal{H}^{N-1}$-almost all points x of $\partial^* E$, where $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure. Moreover, $|D\chi_E|$ coincides with the restriction of $\mathcal{H}^{N-1}$ to $\partial^* E$ [7, 32].

Throughout the text we will use the notation $\{u \geq s\}$ to denote $\{x \in \Omega : u(x) \geq s\}$, $s \in \mathbb{R}$. Also, when we write a.e. without specifying the measure we refer to the Lebesgue measure.

2.2. A generalized Green's formula. Let Ω be an open subset of $\mathbb{R}^N$. Following [9], let

$$X_p(\Omega) := \{z \in L^\infty(\Omega; \mathbb{R}^N) : \operatorname{div} z \in L^p(\Omega)\}.$$

If $z \in X_p(\Omega)$ and $w \in L^q(\Omega) \cap BV(\Omega)$, $p^{-1} + q^{-1} = 1$, we define the functional $z \cdot Dw : C_0^\infty(\Omega) \to \mathbb{R}$ by the formula

$$\langle z \cdot Dw, \varphi \rangle := -\int_\Omega w\, \varphi \operatorname{div} z\, dx - \int_\Omega w\, z \cdot \nabla \varphi\, dx. \tag{2.2}$$

Then $z \cdot Dw$ is a Radon measure in Ω, $\int_\Omega z \cdot Dw\, \varphi = \int_\Omega z \cdot \nabla w \varphi\, dx$ for all $\varphi \in C_c^\infty(\Omega)$, $w \in L^q(\Omega) \cap W^{1,1}(\Omega)$, and

$$\left| \int_B z \cdot Dw \right| \leq \int_B |z \cdot Dw| \leq \|z\|_{L^\infty(\Omega;\mathbb{R}^N)} \int_B |Dw| \qquad \forall \text{ Borel sets } B \subseteq \Omega.$$

We denote by $q_z(Dw) \in L^\infty(\Omega, |Dw|)$ (essentially bounded functions with respect to the measure $|Dw|$) the density of $z \cdot Dw$ with respect to $|Dw|$. This notation is used in the proof of Proposition 3.4. We recall the following result proved in [9].

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary, and let $z \in X_p(\Omega)$. Then there exists a function $[z \cdot \nu^\Omega] \in L^\infty(\partial\Omega)$ such that $\|[z \cdot \nu^\Omega]\|_{L^\infty(\partial\Omega)} \leq \|z\|_{L^\infty(\Omega;\mathbb{R}^N)}$, and, for any $u \in BV(\Omega) \cap L^q(\Omega)$, we have*

$$\int_\Omega u \operatorname{div} z\, dx + \int_\Omega (z \cdot Du) = \int_{\partial\Omega} [z \cdot \nu^\Omega] u\, d\mathcal{H}^{N-1}.$$

Remark 1. Let $\Omega_1, \Omega \subset \mathbb{R}^N$ be two bounded Lipschitz open sets with $\Omega_1 \subset\subset \Omega$, $\Omega_2 = \Omega \setminus \overline{\Omega_1}$, and let $z_1 \in X_p(\Omega_1)$, $z_2 \in X_p(\Omega_2)$. Assume that

$$[z_1 \cdot \nu^{\Omega_1}](x) = -[z_2 \cdot \nu^{\Omega_2}](x) \qquad \text{for } \mathcal{H}^{N-1}\text{-a.e } x \in \partial\Omega_1.$$

Then if we define $z := z_1$ on Ω_1 and $z := z_2$ on Ω_2, we have $z \in X_p(\Omega)$.

3. The total variation with respect to an anisotropy.

3.1. The ϕ-total variation. Let us define the general notion of total variation with respect to a metric integrand. Following [10] we say that a function $\phi : \Omega \times \mathbb{R}^N \to [0, \infty)$ is a *metric integrand* if ϕ is a Borel function satisfying the following conditions:

$$\text{for a.e. } x \in \Omega, \text{ the map } \xi \in \mathbb{R}^N \to \phi(x, \xi) \text{ is convex,} \tag{3.1}$$

$$\phi(x, t\xi) = |t|\phi(x, \xi) \qquad \forall x \in \Omega, \quad \forall \xi \in \mathbb{R}^N, \quad \forall t \in \mathbb{R}, \tag{3.2}$$

and there exists a constant $\Lambda > 0$ such that

$$0 \le \phi(x, \xi) \le \Lambda \|\xi\| \qquad \forall x \in \Omega, \quad \forall \xi \in \mathbb{R}^N. \tag{3.3}$$

We could be more precise and use the term symmetric metric integrand, but for simplicity we use the term metric integrand. Recall that the polar function $\phi^0 : \Omega \times \mathbb{R}^N \to \mathbb{R}$ of ϕ is defined by

$$\phi^0(x, \xi^*) = \sup\{\langle \xi^*, \xi\rangle : \xi \in \mathbb{R}^N, \, \phi(x, \xi) \le 1\}. \tag{3.4}$$

The function $\phi^0(x, \cdot)$ is convex and lower semicontinuous.

For any $p \in [1, \infty]$, we define

$$\mathcal{K}_\phi^{p,D}(\Omega) := \{\sigma \in X_p(\Omega) : \phi^0(x, \sigma(x)) \le 1 \,\text{for a.e. } x \in \Omega\},$$

$$\mathcal{K}_\phi^{p}(\Omega) := \{\sigma \in X_p(\Omega) : \phi^0(x, \sigma(x)) \le 1 \text{ for a.e. } x \in \Omega, \, [\sigma \cdot \nu^\Omega] = 0\},$$

and for any open set $U \subseteq \Omega$, we define

$$\mathcal{K}_\phi^{p,c}(U) := \{\sigma \in X_p(U) : \phi^0(x, \sigma(x)) \le 1 \text{ for a.e. } x \in U, \, \text{spt}(\sigma) \text{ is compact in } U\}.$$

Definition 3.1. *Let $u \in L^1(\Omega)$. We define the ϕ-total variation of u in Ω as*

$$\int_\Omega |Du|_\phi := \sup\left\{ \int_\Omega u \, \text{div}\, \sigma \, dx : \sigma \in \mathcal{K}_\phi^\infty(\Omega)\right\}. \tag{3.5}$$

We set $BV_\phi(\Omega) := \{u \in L^1(\Omega) : \int_\Omega |Du|_\phi < \infty\}$, which is a Banach space when endowed with the norm $|u|_{BV_\phi(\Omega)} := \int_\Omega |u| dx + \int_\Omega |Du|_\phi$.

We say that $E \subseteq \mathbb{R}^N$ has finite ϕ-perimeter in Ω if $\chi_E \in BV_\phi(\Omega)$. We set

$$P_\phi(E, \Omega) := \int_\Omega |D\chi_E|_\phi.$$

If $\Omega = \mathbb{R}^N$, we denote $P_\phi(E) := P_\phi(E, \mathbb{R}^N)$. By assumption (3.3), if $E \subseteq \mathbb{R}^N$ has finite perimeter in Ω, it also has finite ϕ-perimeter in Ω.

Remark 2. Notice that the definition of $\int_\Omega |Du|_\phi$ is slightly different from those given in [5] and [10]. In [5] the vector fields σ are in $\mathcal{K}_\phi^{N,c}(\Omega)$. In [10] they are such that $\text{div}\, \sigma \in L^\infty(\Omega)$, but the authors consider the case of bounded variation functions with respect to a Radon measure on $\mathbb{R}^N$. For convenience in proving Lemma 3.8, we have chosen Definition 3.1. In Proposition 3.4 we shall prove that our definition is equivalent to the one given in [5].

The coarea formula for the ϕ-total variation was proved in [10] (see also [5] in a slightly different formulation):

$$\int_\Omega |Du|_\phi = \int_{\mathbb{R}} P_\phi(\{u > s\}, \Omega)\, ds \qquad \forall u \in BV_\phi(\Omega). \tag{3.6}$$

Moreover, if T is a Lipschitz function and $u \in BV_\phi(\Omega)$, then $T(u) \in BV_\phi(\Omega)$ [10].

If $u \in BV_\phi(\Omega)$, then u determines a Radon measure in Ω. Indeed, for each open set $U \subseteq \Omega$ we define

$$|Du|_\phi(U) := \sup\left\{\int_U u \operatorname{div}\sigma\ dx : \sigma \in \mathcal{K}_\phi^{\infty,c}(U)\right\}. \tag{3.7}$$

Notice that $|Du|_\phi(U) \leq \int_U |Du|_\phi$ for any open set $U \subseteq \Omega$ with Lipschitz boundary. We have that $|Du|_\phi(U)$ is an inner content (see [36] for the definition and Appendix A.1 for the proof). Let

$$\mu^*(E) := \inf\{|Du|_\phi(U) : U \text{ is an open set in } \Omega,\ E \subseteq U\}$$

be the outer measure induced by $|Du|_\phi$. Then for any Borel set F we define $\mu(F) = \mu^*(F)$. Then μ is a regular Borel measure [36, p. 235]. We shall write $|Du|_\phi(E)$ instead of $\mu(E)$.

Definition 3.2. *Let $\phi : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a metric integrand, $B \subseteq \Omega$. We say that ϕ is coercive in B if there exist $\beta \geq \alpha > 0$ such that*

$$\alpha\|\xi\| \leq \phi(x,\xi) \leq \beta\|\xi\| \qquad \forall x \in B, \quad \forall \xi \in \mathbb{R}^N. \tag{3.8}$$

We say that ϕ is continuous in B if ϕ restricted to $B \times \mathbb{R}^N$ is a continuous function of (x,ξ). If $B = \Omega$, we just say that ϕ is coercive (resp., continuous). We say that ϕ is coercive (resp., continuous) near $\partial\Omega$ if there exists $\Omega_1 \subset\subset \Omega$, an open bounded set with Lipschitz boundary such that ϕ is coercive (resp., continuous) in a neighborhood of $\Omega \setminus \Omega_1$.

Lemma 3.3. *Assume that $B \subset \Omega$ is an open set with Lipschitz boundary and that the metric integrand $\phi : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is coercive in B. Then $u \in BV(B)$ and $\int_B |Du| \leq \frac{1}{\alpha}\int_\Omega |Du|_\phi$, where α is the coercivity constant of ϕ in B.*

Proof. Assume that $\phi(x,\xi) \geq \alpha|\xi| =: \phi_B(x,\xi)$ for any $x \in B$, $\xi \in \mathbb{R}^N$. Since $\phi^0(x,\xi^*) \leq \phi_B^0(x,\xi^*)$ for any $x \in B$, $\xi^* \in \mathbb{R}^N$, we have

$$\begin{aligned}\int_\Omega |Du|_\phi = \sup\left\{\int_\Omega u \operatorname{div}\sigma\ dx : \sigma \in \mathcal{K}_\phi^\infty(\Omega)\right\} &\geq \sup\left\{\int_\Omega u \operatorname{div}\sigma\ dx : \sigma \in \mathcal{K}_\phi^{\infty,c}(B)\right\}\\ \geq \sup\left\{\int_\Omega u \operatorname{div}\sigma\ dx : \sigma \in \mathcal{K}_{\phi_B}^{\infty,c}(B)\right\} &= \alpha\int_B |Du|. \qquad \blacksquare\end{aligned}$$

Proposition 3.4. *Let Ω be a bounded open set in $\mathbb{R}^N$ with Lipschitz boundary. Let $\phi : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a metric integrand. Assume that ϕ is coercive and $u \in BV(\Omega)$. Then*

$$\int_\Omega |Du|_\phi = \sup\left\{\int_\Omega u \operatorname{div}\sigma\ dx : \sigma \in \mathcal{K}_\phi^{\infty,c}(\Omega)\right\}.$$

Assume that ϕ is continuous and coercive. Then

$$\int_\Omega |Du|_\phi = \int_\Omega \phi(x,\nabla u)\, dx \qquad \forall u \in W^{1,1}(\Omega). \tag{3.9}$$

Proof. Let us first prove that

$$\int_\Omega |Du|_\phi = \int_\Omega Q(x)|Du|, \tag{3.10}$$

with $Q(x) := \operatorname{esssup}_{\sigma\in\mathcal{K}_\phi^\infty(\Omega)} q_\sigma(Du)(x)$, where the essential supremum is taken with respect to the measure $|Du|$ and q_σ is the density of $\sigma\cdot Du$ with respect to $|Du|$ (see section 2.2). The proof of (3.10) mimics the proof of Theorem 4.3 in [5]. Using the integration by parts formula (2.1), we have

$$\int_\Omega |Du|_\phi = \sup_{\sigma\in\mathcal{K}_\phi^\infty(\Omega)} \int_\Omega \sigma\cdot Du = \sup_{\sigma\in\mathcal{K}_\phi^\infty(\Omega)} \int_\Omega q_\sigma(Du)|Du|.$$

Let $L^1(\Omega,|Du|)$ be the space of functions whose absolute value is integrable with respect to the measure $|Du|$, and let $T_u : \mathcal{K}_\phi^\infty(\Omega) \to L^1(\Omega,|Du|)$ be the operator defined by $T_u(\sigma)(x) = -q_\sigma(Du)(x)$ for $|Du|$-almost every $x\in\Omega$. Let

$$H = \{T_u(\sigma) : \sigma\in\mathcal{K}_\phi^\infty(\Omega)\}.$$

We prove that the set H is C^1-inf-stable. For that we have to prove that if $\{\sigma_i\}_{i\in I}$ is a finite family of elements of $\mathcal{K}_\phi^\infty(\Omega)$ and $\{\alpha_i\}_{i\in I}$ is a finite family of nonnegative functions of $C^1(\overline{\Omega})$ such that $\sum_{i\in I}\alpha_i = 1$ in Ω, then

$$\sum_{i\in I}\alpha_i T_u(\sigma_i) \ge T_u(\sigma), \qquad |Du|\text{-a.e. in } \Omega, \tag{3.11}$$

for some $\sigma\in\mathcal{K}_\phi^\infty(\Omega)$. It suffices to take $\sigma := \sum_{i\in I}\alpha_i\sigma_i$. By the convexity of ϕ^0 it follows that $\sigma\in\mathcal{K}_\phi^\infty(\Omega)$. Moreover, by [11, Remark 1.5], we get (3.11). This proves that H is C^1-inf-stable.

Then by [11, Lemma 4.3], we have

$$\inf_{\sigma\in\mathcal{K}_\phi^\infty(\Omega)} \int_\Omega T_u(\sigma)|Du| = \inf_{\sigma\in\mathcal{K}_\phi^\infty(\Omega)} \int_\Omega -q_\sigma(Du)(x)|Du| = \int_\Omega -Q(x)|Du|.$$

This implies (3.10).

Now, the equalities

$$\sup_{\sigma\in\mathcal{K}_\phi^N(\Omega)} q_\sigma(Du)(x) = \sup_{\sigma\in\mathcal{K}_\phi^{N,c}(\Omega)} q_\sigma(Du)(x) = \sup_{\sigma\in C_c^1(\Omega)} \int_\Omega u \operatorname{div}\sigma\, dx \tag{3.12}$$

follow from Proposition 3.2 in [5] and

$$\sup_{\sigma\in\mathcal{K}_\phi^{N,c}(\Omega)} q_\sigma(Du)(x) \le \sup_{\sigma\in\mathcal{K}_\phi^N(\Omega)} q_\sigma(Du)(x) \le \sup_{\sigma\in\mathcal{K}_\phi^{N,D}(\Omega)} q_\sigma(Du)(x) = \sup_{\sigma\in\mathcal{K}_\phi^{N,c}(\Omega)} q_\sigma(Du)(x),$$

where the first two inequalities are obvious and the last equality follows from Lemma 4.5 in [5]. Then, using (3.10) and (3.12), we have

$$\begin{aligned}\sup_{\sigma\in\mathcal{K}_\phi^{\infty,c}(\Omega)}\int_\Omega u\,\mathrm{div}\,\sigma\,dx &\le \int_\Omega |Du|_\phi = \int_\Omega Q(x)|Du| \le \int_\Omega \sup_{\sigma\in\mathcal{K}_\phi^{N}(\Omega)} q_\sigma(Du)(x)|Du|\\ &= \int_\Omega \sup_{\sigma\in\mathcal{K}_\phi^{N,c}(\Omega)} q_\sigma(Du)(x)|Du| = \sup_{\sigma\in C_c^1(\Omega)}\int_\Omega u\,\mathrm{div}\,\sigma\,dx\\ &\le \sup_{\sigma\in\mathcal{K}_\phi^{\infty,c}(\Omega)}\int_\Omega u\,\mathrm{div}\,\sigma\,dx.\end{aligned}$$

Finally, the identity (3.9) is stated in [5, Theorem 5.1]. ■

Example. An interesting case occurs when $g:\Omega\to[0,\infty)$ is a bounded Borel function. Let $\phi(x,\xi)=g(x)|\xi|$. Then [5]

$$\phi^0(x,\xi^*) := \begin{cases} 0 & \text{if} \quad g(x)=0,\ \xi^*=0,\\ +\infty & \text{if} \quad g(x)=0,\ \xi^*\neq 0,\\ \frac{|\xi^*|}{g(x)} & \text{if} \quad \xi^*\in\mathbb{R}^N,\ g(x)>0. \end{cases} \tag{3.13}$$

If $\sigma\in X_\infty(\Omega)$ and $\phi^0(x,\sigma(x))\le 1$, then we may write $\sigma(x)=g(x)z(x)$, where $z\in L^\infty(\Omega;\mathbb{R}^N)$ is such that $|z(x)|\le 1$ for a.e. $x\in\Omega$, and

$$\int_\Omega g|Du| := \sup\left\{\int_\Omega u\,\mathrm{div}\,(gz)\,dx : gz\in X_\infty(\Omega),\ |z(x)|\le 1 \quad \text{for a.e. } x\in\Omega\right\}.$$

3.2. Lower semicontinuity and relaxation results. From Definition 3.1, it follows that $u\in L^1(\Omega)\to\int_\Omega|Du|_\phi$ and $E\to P_\phi(E,\Omega)$ are lower semicontinuous with respect to the L^1 convergence. The following result was proved in [10] when $\Omega=\mathbb{R}^N$. The proof adapts easily.

Proposition 3.5. *Assume that $\phi:\Omega\times\mathbb{R}^N\to[0,\infty)$ is a metric integrand. Let*

$$J(u) := \begin{cases} \displaystyle\int_\Omega \phi(x,\nabla u)\,dx & \text{if} \quad u\in W^{1,1}(\Omega),\\ +\infty & \text{if} \quad u\in L^1(\Omega)\setminus W^{1,1}(\Omega).\end{cases}$$

Let $\overline{J}$ be the relaxed functional, that is,

$$\overline{J}(u) := \inf\{\liminf_n J(u_n) : u_n\to u \text{ in } L^1(\Omega),\ u_n\in W^{1,1}(\Omega)\}.$$

Then for every $u\in BV_\phi(\Omega)$, we have $\overline{J}(u)=\int_\Omega|Du|_\phi$. Hence, for any $u\in BV_\phi(\Omega)$, there exists a sequence $u_n\in W^{1,1}(\Omega)$ such that $\int_\Omega\phi(x,\nabla u_n)\to\int_\Omega|Du|_\phi$. In particular, $BV_\phi(\Omega)$ is the finiteness domain of $\overline{J}$.

Our main purpose in the rest of this section is to prove the following lower semicontinuity result.

Theorem 3.6. *Assume that ϕ satisfies the assumptions of Proposition 3.11. Let $\varphi \in L^1(\partial\Omega)$,*

$$J_{\phi,\varphi}(u) := \begin{cases} \displaystyle\int_\Omega \phi(x, \nabla u) & \text{if } u \in W^{1,1}(\Omega) \text{ and } u = \varphi \text{ on } \partial\Omega, \\ +\infty & \text{otherwise,} \end{cases} \tag{3.14}$$

and

$$\mathcal{J}_{\phi,\varphi}(u) := \begin{cases} \displaystyle\int_\Omega |Du|_\phi + \int_{\partial\Omega} \phi(x, \nu^\Omega)|u - \varphi| \, d\mathcal{H}^{N-1} & \text{if } u \in BV_\phi(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.15}$$

Then the functional $\mathcal{J}_{\phi,\varphi}(u)$ is the relaxed functional of $J_{\phi,\varphi}(u)$.

The proof of this theorem will be the consequence of a series of lemmas and propositions below.

Lemma 3.7. *Assume that $\phi : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a metric integrand which is continuous and coercive in Ω. Let $u \in BV_\phi(\Omega)$, $\varphi, \varphi_n \in L^1(\partial\Omega)$, such that $\varphi_n \to \varphi$ in $L^1(\partial\Omega)$. Then there exists $u_n \in W^{1,1}(\Omega)$, $u_n|_{\partial\Omega} = \varphi_n$, such that $u_n \to u$ in $L^1(\Omega)$ and*

$$\int_\Omega \phi(x, \nabla u_n) \, dx \to \int_\Omega |Du|_\phi + \int_{\partial\Omega} \phi(x, \nu^\Omega)|u - \varphi| \, d\mathcal{H}^{N-1}.$$

Proof. By Theorem 6 in [49], there exists $v_n \in W^{1,1}(\Omega)$, $v_n|_{\partial\Omega} = \varphi$, such that $v_n \to u$ in $L^1(\Omega)$ and

$$\int_\Omega \phi(x, \nabla v_n) \, dx \to \int_\Omega |Du|_\phi + \int_{\partial\Omega} \phi(x, \nu^\Omega)|u - \varphi| \, d\mathcal{H}^{N-1}.$$

Now, by a standard result in the theory of Sobolev spaces [35], there exists $w_n \in W^{1,1}(\Omega)$, $w_n|_{\partial\Omega} = \varphi_n - \varphi$, such that $w_n = 0$ if $\operatorname{dist}(x, \partial\Omega) > \frac{1}{n}$ and

$$\int_\Omega |\nabla w_n| \, dx \leq \int_{\partial\Omega} |\varphi_n - \varphi| \, d\mathcal{H}^{N-1} + \frac{1}{n}.$$

The function $u_n = v_n + w_n$ satisfies the lemma. ∎

Lemma 3.8. *Assume that Ω_1, Ω are as in Definition 3.2 and ϕ is coercive in a neighborhood of $\Omega \setminus \Omega_1$. Let $u \in BV_\phi(\Omega)$. Assume that $|Du|(\partial\Omega_1) = 0$. Then*

$$\int_\Omega |Du|_\phi = \int_{\Omega_1} |Du|_\phi + \int_{\Omega\setminus\overline{\Omega_1}} |Du|_\phi.$$

Proof. Since by Lemma 3.3 we know that $u \in BV(U)$ for some neighborhood U of $\Omega \setminus \Omega_1$, then $u \in BV(U)$ and, by a slight perturbation of Ω_1, we may assume that $|Du|(\partial\Omega_1) = 0$. Given $\epsilon > 0$, let φ be a smooth function with support in Ω_1 and such that $\varphi(x) = 1$ for all $x \in \Omega_{1,\epsilon} := \{x \in \Omega_1 : \operatorname{dist}(x, \partial\Omega_1) > \epsilon\}$. Now, if $\sigma \in \mathcal{K}_\phi^\infty(\Omega)$, then

$$\int_\Omega u \operatorname{div} \sigma \, dx = \int_{\Omega_1} u \operatorname{div}(\varphi\sigma) \, dx + \int_{\Omega\setminus\overline{\Omega_{1,\epsilon}}} u \operatorname{div}((1-\varphi)\sigma) \, dx \leq \int_{\Omega_1} |Du|_\phi + \int_{\Omega\setminus\overline{\Omega_{1,\epsilon}}} |Du|_\phi. \tag{3.16}$$

Since $|Du|(\partial\Omega_1) = 0$ and $u \in BV(\Omega \setminus \overline{\Omega_{1,\epsilon}})$ for ϵ small enough,

$$\int_{\Omega\setminus\overline{\Omega_{1,\epsilon}}} |Du|_\phi \to \int_{\Omega\setminus\overline{\Omega_1}} |Du|_\phi \qquad \text{as } \epsilon \to 0+.$$

Taking suprema in $\sigma \in \mathcal{K}_\phi^\infty(\Omega)$ in (3.16), we obtain

$$\int_\Omega |Du|_\phi \le \int_{\Omega_1} |Du|_\phi + \int_{\Omega\setminus\overline{\Omega_1}} |Du|_\phi.$$

Let $\Omega_2 = \Omega \setminus \overline{\Omega_1}$. Observe that if $\sigma_i \in \mathcal{K}_\phi^\infty(\Omega_i)$, then $[\sigma_i \cdot \nu^{\Omega_1}] = 0$, $i = 1, 2$. Thus, if we define

$$\sigma(x) := \begin{cases} \sigma_1(x) & \text{if} \quad x \in \Omega_1, \\ \sigma_2(x) & \text{if} \quad x \in \Omega_2, \end{cases} \tag{3.17}$$

then $\sigma \in \mathcal{K}_\phi^\infty(\Omega)$ (see Remark 1). Since

$$\int_{\Omega_1} u \operatorname{div} \sigma_1 \, dx + \int_{\Omega_2} u \operatorname{div} \sigma_2 \, dx = \int_\Omega u \operatorname{div} \sigma \, dx \le \int_\Omega |Du|_\phi,$$

taking suprema in $\sigma_1 \in \mathcal{K}_\phi^\infty(\Omega_1)$, $\sigma_2 \in \mathcal{K}_\phi^\infty(\Omega_2)$, we obtain

$$\int_{\Omega_1} |Du|_\phi + \int_{\Omega_2} |Du|_\phi \le \int_\Omega |Du|_\phi. \qquad \blacksquare$$

Lemma 3.9. *Assume that $\phi : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a metric integrand which admits an extension as a metric integrand to an open bounded set Q with Lipschitz boundary such that $\Omega \subset\subset Q$. Assume that the extension is continuous and coercive in a neighborhood of $Q \setminus \Omega$. Let $u \in BV_\phi(\Omega)$, $\tilde\varphi \in W^{1,1}(Q \setminus \Omega)$ be such that $\tilde\varphi|_{\partial\Omega} = \varphi \in L^1(\partial\Omega)$. Let*

$$\tilde u(x) := \begin{cases} u(x) & \text{if} \quad x \in \Omega, \\ \tilde\varphi & \text{if} \quad x \in Q \setminus \Omega. \end{cases} \tag{3.18}$$

Then $u \in BV_\phi(Q)$ and

$$\int_Q |D\tilde u|_\phi = \int_\Omega |Du|_\phi + \int_{Q\setminus\overline{\Omega}} |\nabla\tilde\varphi|_\phi + \int_{\partial\Omega} \phi(x, \nu^\Omega)|u - \varphi| \, d\mathcal{H}^{N-1}. \tag{3.19}$$

Proof. Let $\Omega_1 \subset\subset \Omega$ be such that Ω_1 has a Lipschitz boundary and ϕ is coercive and continuous in a neighborhood U of $Q \setminus \Omega_1$. Since $\tilde u \in BV(U)$, by a slight perturbation of Ω_1, we may assume that $|D\tilde u|(\partial\Omega_1) = 0$. By this and Lemma 3.8 we have

$$\int_Q |D\tilde u|_\phi = \int_{\Omega_1} |D\tilde u|_\phi + \int_{Q\setminus\Omega_1} |D\tilde u|_\phi = \int_{\Omega_1} |Du|_\phi + \int_{Q\setminus\Omega_1} |D\tilde u|_\phi.$$

By Proposition 3.4 and Theorem 7 of [49] we know that

$$\int_{Q\setminus\Omega_1} |D\tilde{u}|_\phi = \int_{\Omega\setminus\Omega_1} |Du|_\phi + \int_{Q\setminus\overline{\Omega}} |\nabla\tilde{\varphi}|_\phi + \int_{\partial\Omega} \phi(x,\nu^\Omega)|u-\varphi|\,d\mathcal{H}^{N-1}.$$

Hence

$$\int_Q |D\tilde{u}|_\phi = \int_{\Omega_1} |Du|_\phi + \int_{\Omega\setminus\Omega_1} |Du|_\phi + \int_{Q\setminus\overline{\Omega}} |\nabla\tilde{\varphi}|_\phi + \int_{\partial\Omega} \phi(x,\nu^\Omega)|u-\varphi|\,d\mathcal{H}^{N-1}.$$

Now, applying again Lemma 3.8, we obtain (3.19). ∎

Proposition 3.10. *Let Ω be an open bounded set in $\mathbb{R}^N$ with Lipschitz boundary. Let $\phi : \Omega\times\mathbb{R}^N \to \mathbb{R}$ be a metric integrand such that ϕ is continuous and coercive in a neighborhood of $\partial\Omega$. Let $u \in BV_\phi(\Omega)$, $\varphi \in L^1(\partial\Omega)$. Then there exists $u_n \in W^{1,1}(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$, $u_n|_{\partial\Omega} = \varphi$, such that*

$$\int_\Omega \phi(x,\nabla u_n)\,dx \to \int_\Omega |Du|_\phi + \int_{\partial\Omega} \phi(x,\nu^\Omega)|u-\varphi|\,d\mathcal{H}^{N-1}.$$

Proof. Since $u \in BV_\phi(\Omega)$, let $u_n \in W^{1,1}(\Omega)$ be such that $u_n \to u$ in $L^1(\Omega)$ and

$$\int_\Omega \phi(x,\nabla u_n) \to \int_\Omega |Du|_\phi. \tag{3.20}$$

Let $\Omega_1 \subset\subset \Omega$ be an open set with Lipschitz boundary such that ϕ is continuous and coercive in a neighborhood of $\Omega\setminus\Omega_1$ and $|Du|(\partial\Omega_1) = 0$. Let $w_1 = u|_{\Omega_1}$, $w_2 = u|_{\Omega\setminus\Omega_1}$. Then by Lemma 3.8 we have

$$\begin{aligned}\int_\Omega |Du|_\phi &= \int_{\Omega_1} |Dw_1|_\phi + \int_{\Omega\setminus\overline{\Omega}_1} |Dw_2|_\phi \le \liminf_n \int_{\Omega_1} \phi(x,\nabla u_n) + \liminf_n \int_{\Omega\setminus\overline{\Omega}_1} \phi(x,\nabla u_n)\\ &\le \liminf_n \int_\Omega \phi(x,\nabla u_n) = \int_\Omega |Du|_\phi.\end{aligned}$$

Then

$$\liminf_n \int_{\Omega_1} \phi(x,\nabla u_n) = \int_{\Omega_1} |Dw_1|_\phi \quad \text{and} \quad \liminf_n \int_{\Omega\setminus\overline{\Omega}_1} \phi(x,\nabla u_n) = \int_{\Omega\setminus\overline{\Omega}_1} |Dw_2|_\phi.$$

By extracting a subsequence, we may assume that

$$\lim_n \int_{\Omega_1} \phi(x,\nabla u_n) = \int_{\Omega_1} |Dw_1|_\phi.$$

Using (3.20), we also have that

$$\lim_n \int_{\Omega\setminus\overline{\Omega}_1} \phi(x,\nabla u_n) = \int_{\Omega\setminus\overline{\Omega}_1} |Dw_2|_\phi.$$

Since ϕ is continuous and coercive in a neighborhood of $\Omega \setminus \Omega_1$, by Reshetnyak's theorem [34], from the last convergence we have that $u_n|_{\partial\Omega_1} \to w_2|_{\partial\Omega_1} = u|_{\partial\Omega_1}$ (since $|Du|(\partial\Omega_1) = 0$).

Now, we apply Lemma 3.7 and find $\tilde{u}_n \in W^{1,1}(\Omega \setminus \overline{\Omega}_1)$ with $\tilde{u}_n|_{\partial\Omega_1} = u_n|_{\partial\Omega_1}$, $\tilde{u}_n|_{\partial\Omega} = \varphi$, and $\tilde{u}_n \to w_2 = u$ in $L^1(\Omega \setminus \overline{\Omega}_1)$, such that

$$\int_{\Omega\setminus\overline{\Omega}_1} \phi(x, \nabla\tilde{u}_n) \to \int_{\Omega\setminus\overline{\Omega}_1} |Du|_\phi + \int_{\partial\Omega} \phi(x, \nu^\Omega)|u - \varphi|\, d\mathcal{H}^{N-1}.$$

Redefine u_n as

$$u_n(x) := \begin{cases} u_n(x) & \text{if} \quad x \in \Omega_1, \\ \tilde{u}_n & \text{if} \quad x \in \Omega \setminus \overline{\Omega_1}. \end{cases} \tag{3.21}$$

Then $u_n \to u$ in $L^1(\Omega)$, $u_n|_{\partial\Omega} = \varphi$, and

$$\begin{aligned}\int_\Omega \phi(x, \nabla u_n) = \int_{\Omega_1} \phi(x, \nabla u_n) + \int_{\Omega\setminus\overline{\Omega_1}} \phi(x, \nabla u_n) \to \int_{\Omega_1} |Du|_\phi + \int_{\Omega\setminus\overline{\Omega}_1} |Du|_\phi \\ + \int_{\partial\Omega} \phi(x, \nu^\Omega)|u - \varphi|\, d\mathcal{H}^{N-1} = \int_\Omega |Du|_\phi + \int_{\partial\Omega} \phi(x, \nu^\Omega)|u - \varphi|\, d\mathcal{H}^{N-1},\end{aligned}$$

where the last identity is given by Lemma 3.8. ∎

Proposition 3.11. *Let Ω, Q be open bounded sets in $\mathbb{R}^N$ with Lipschitz boundary such that $\Omega \subset\subset Q$. Let $\phi : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a metric integrand which admits an extension to $Q \times \mathbb{R}^N$ such that ϕ is continuous and coercive in a neighborhood of $Q \setminus \Omega$. Let $u \in BV_\phi(\Omega)$, $\varphi \in L^1(\partial\Omega)$. Let $u_n \in W^{1,1}(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$. Then*

$$\int_\Omega |Du|_\phi + \int_{\partial\Omega} \phi(x, \nu^\Omega)|u - \varphi|\, d\mathcal{H}^{N-1} \le \liminf_n \int_\Omega \phi(x, \nabla u_n)\, dx + \int_{\partial\Omega} \phi(x, \nu^\Omega)|u_n - \varphi|\, d\mathcal{H}^{N-1}. \tag{3.22}$$

Proof. Let $\tilde{\varphi} \in W^{1,1}(Q\setminus\overline{\Omega})$ be such that $\tilde{\varphi}|_{\partial\Omega} = \varphi$. Let Ω_1 be as usual so that $|Du|(\partial\Omega_1) = 0$. Let $\tilde{u}$, $\tilde{u}_n$ be the extensions defined as in Lemma 3.9. Then, by Proposition 3.5 and the lower semicontinuity of the ϕ-total variation, we have

$$\int_{\Omega_1} |D\tilde{u}|_\phi \le \liminf_n \int_{\Omega_1} \phi(x, \nabla u_n),$$

$$\int_{Q\setminus\overline{\Omega}_1} |D\tilde{u}|_\phi \le \liminf_n \int_{Q\setminus\overline{\Omega}_1} |D\tilde{u}_n|_\phi.$$

Using Lemma 3.9 and Proposition 3.4, we may write

$$\begin{aligned}\int_{Q\setminus\overline{\Omega}_1} |D\tilde{u}|_\phi &\le \liminf_n \left\{ \int_{\Omega\setminus\overline{\Omega_1}} |Du_n|_\phi + \int_{Q\setminus\overline{\Omega}} |\nabla\tilde{\varphi}|_\phi + \int_{\partial\Omega} \phi(x, \nu^\Omega)|u_n - \varphi|\, d\mathcal{H}^{N-1} \right\} \\ &= \liminf_n \left\{ \int_{\Omega\setminus\overline{\Omega_1}} \phi(x, \nabla u_n) + \int_{Q\setminus\overline{\Omega}} |\nabla\tilde{\varphi}|_\phi + \int_{\partial\Omega} \phi(x, \nu^\Omega)|u_n - \varphi|\, d\mathcal{H}^{N-1} \right\}.\end{aligned}$$

Now, using Lemma 3.8, we have

$$
\begin{aligned}
&\int_Q |D\tilde u|_\phi = \int_{\Omega_1} |D\tilde u|_\phi + \int_{Q\setminus\overline{\Omega}_1} |D\tilde u|_\phi \le \liminf_n \int_{\Omega_1} \phi(x,\nabla u_n)\\
&\quad + \liminf_n \left\{ \int_{\Omega\setminus\overline{\Omega_1}} \phi(x,\nabla u_n) + \int_{Q\setminus\overline{\Omega}} |\nabla\tilde\varphi|_\phi + \int_{\partial\Omega} \phi(x,\nu^\Omega)|u_n-\varphi|\,d\mathcal{H}^{N-1}\right\}\\
&\le \liminf_n \left\{ \int_{\Omega_1} \phi(x,\nabla u_n) + \int_{\Omega\setminus\overline{\Omega_1}} \phi(x,\nabla u_n) + \int_{Q\setminus\overline{\Omega}} |\nabla\tilde\varphi|_\phi + \int_{\partial\Omega} \phi(x,\nu^\Omega)|u_n-\varphi|\,d\mathcal{H}^{N-1}\right\}\\
&= \liminf_n \left\{ \int_{\Omega} \phi(x,\nabla u_n) + \int_{Q\setminus\overline{\Omega}} |\nabla\tilde\varphi|_\phi + \int_{\partial\Omega} \phi(x,\nu^\Omega)|u_n-\varphi|\,d\mathcal{H}^{N-1}\right\}.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\int_{\Omega_1} |D\tilde u|_\phi + \int_{Q\setminus\overline{\Omega}_1} |D\tilde u|_\phi = \int_{\Omega_1} |D\tilde u|_\phi + \int_{\Omega\setminus\overline{\Omega}_1} |D\tilde u|_\phi + \int_{Q\setminus\overline{\Omega}} |\nabla\tilde\varphi|_\phi\\
&\quad + \int_{\partial\Omega} \phi(x,\nu^\Omega)|u-\varphi|\,d\mathcal{H}^{N-1} = \int_\Omega |Du|_\phi + \int_{Q\setminus\overline{\Omega}} |\nabla\tilde\varphi|_\phi + \int_{\partial\Omega} \phi(x,\nu^\Omega)|u-\varphi|\,d\mathcal{H}^{N-1},
\end{aligned}
$$

comparing the last two expressions, we obtain (3.22). ■

Observe that Theorem 3.6 follows from Propositions 3.10 and 3.11.

4. An extension of Green's formulas. Throughout this section we assume that Ω is a bounded open set with Lipschitz boundary. We also assume that $\phi : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a metric integrand. We just give an overview of the main results here and provide a sketch of their proofs in Appendix A.2.

4.1. The measure $z \cdot Du$. Let $u \in BV_\phi(\Omega)$ and $z \in X_\infty(\Omega)$. We define the functional $z \cdot Du : \mathcal{D}(\Omega) \to \mathbb{R}$ as in (2.2). Although a more general functional setting is possible, we shall restrict our discussion to this case.

Let us write

$$\mathcal{A}_\infty(\Omega) := \{z \in X_\infty(\Omega) : \|\phi^0(x,z(x))\|_{L^\infty(\Omega)} < \infty\}. \tag{4.1}$$

To develop the theory we shall assume from now on that $z \in \mathcal{A}_\infty(\Omega)$.

Proposition 4.1. *For any open set $U \subset \Omega$ and for any function $\varphi \in \mathcal{D}(U)$, one has*

$$|\langle z \cdot Du, \varphi\rangle| \le \|\varphi\|_\infty \|\phi^0(x,z)\|_{L^\infty(U)} |Du|_\phi(U). \tag{4.2}$$

In particular, $z \cdot Du$ is a Radon measure in Ω.

Lemma 4.2. *Let $u \in BV_\phi(\Omega)$, $\sigma \in \mathcal{A}_\infty(\Omega)$ with $\|\phi^0(x,z(x))\|_{L^\infty(\Omega)} \le 1$. Assume that $\int_\Omega \sigma \cdot Du = \int_\Omega |Du|_\phi$. Then for any $b \in \mathbb{R}$ we have*

$$\int_\Omega \sigma \cdot D(u-b)^+ = \int_\Omega |D(u-b)^+|_\phi \quad \text{and} \quad \int_\Omega \sigma \cdot D(u\wedge b) = \int_\Omega |D(u\wedge b)|_\phi$$

where $(u-b)^+ = \max(u-b,0)$ and $u \wedge b = \inf(u,b)$.

Proof. By the observation following (3.6) we know that $(u-b)^+, u \wedge b \in BV_\phi(\Omega)$. Then we have

$$\begin{aligned}\int_\Omega |Du|_\phi = \int_\Omega \sigma\cdot Du = \int_\Omega \sigma \cdot D(u-b)^+ + \int_\Omega \sigma\cdot D(u\wedge b)\\ \le \int_\Omega |D(u-b)^+|_\phi + \int_\Omega |D(u\wedge b)|_\phi = \int_\Omega |Du|_\phi,\end{aligned}$$

where the inequality follows from Proposition 4.1 and the last equality follows from the coarea formula (3.6). Lemma 4.2 follows. ■

4.2. Traces. The following result can be proved as in [9] (see also [8]).

Proposition 4.3. *Assume that ϕ is continuous and coercive at $\partial\Omega$. There exists a bilinear map $\langle z, u\rangle_{\partial\Omega} : \mathcal{A}_\infty(\Omega)\times BV_\phi(\Omega) \to \mathbb{R}$ such that*

$$\langle z,u\rangle_{\partial\Omega} = \int_{\partial\Omega} u(x) z(x)\cdot\nu(x)\; d\mathcal{H}^{N-1} \qquad \text{if} \quad z\in C^1(\Omega,\mathbb{R}^N)\cap C(\overline{\Omega},\mathbb{R}^N), \tag{4.3}$$

$$|\langle z,u\rangle_{\partial\Omega}| \le \|z\|_{L^\infty(\Omega;\mathbb{R}^N)} \int_{\partial\Omega} |u(x)|\; d\mathcal{H}^{N-1} \qquad \forall\, z, u. \tag{4.4}$$

Proposition 4.4. *Assume that ϕ is continuous and coercive at $\partial\Omega$. Then there exists a linear operator $\gamma : \mathcal{A}_\infty(\Omega) \to L^\infty(\partial\Omega)$ such that*

$$\|\gamma(z)\|_{L^\infty(\partial\Omega)} \le \|z\|_{L^\infty(\Omega;\mathbb{R}^N)}, \tag{4.5}$$

$$\langle z,u\rangle_{\partial\Omega} = \int_{\partial\Omega} \gamma(z)(x)u(x)\, d\mathcal{H}^{N-1} \qquad \forall\, u\in BV_\phi(\Omega), \tag{4.6}$$

$$\gamma(z)(x) = z(x)\cdot\nu(x) \qquad \forall\, x\in\partial\Omega \;\; \text{if} \;\; z\in C^1(\overline{\Omega},\mathbb{R}^N). \tag{4.7}$$

The function $\gamma(z)$ is a weakly defined trace on $\partial\Omega$ of the normal component of z. We shall denote $\gamma(z)$ by $[z\cdot\nu]$.

Proof. Fix $z\in\mathcal{A}_\infty(\Omega)$, $u\in BV_\phi(\Omega)$. Consider the functional $F : L^\infty(\partial\Omega)\to\mathbb{R}$ defined by $F(u) := \langle z, w\rangle_{\partial\Omega}$, where $w\in BV_\phi(\Omega)$ is such that $w|_{\partial\Omega} = u|_{\partial\Omega}$. By estimate (4.4), we have $|F(u)| \le \|z\|_{L^\infty(\Omega;\mathbb{R}^N)} \int_{\partial\Omega}|u|$. Hence there exists a function $\gamma(z)\in L^\infty(\partial\Omega)$ such that

$$F(u) = \int_{\partial\Omega} \gamma(z)(x)u(x)\, d\mathcal{H}^{N-1},$$

and the result follows. ■

4.3. Green's formula. We now give the expected *Green's formula* relating the function $[z\cdot\nu]$ and the measure $z\cdot Du$.

Theorem 4.5. *Assume that ϕ is continuous and coercive at $\partial\Omega$. Let $z\in\mathcal{A}_\infty(\Omega)$, $u\in BV_\phi(\Omega)$. Then*

$$\int_\Omega u\,\mathrm{div}(z)\,dx + \int_\Omega z\cdot Du = \int_{\partial\Omega} [z\cdot\nu]u\, d\mathcal{H}^{N-1}. \tag{4.8}$$

Proof. By Proposition 3.10, there exists a sequence $u_n \in W^{1,1}(\Omega)$ converging to u in $L^1(\Omega)$ such that $u_n|_{\partial\Omega} = u|_{\partial\Omega}$ and $\int_\Omega \phi(x, \nabla u_n) \to \int_\Omega |Du|_\phi$. Then, by Lemma A.2, we have

$$\int_\Omega u \operatorname{div}(z)\, dx + \int_\Omega z \cdot Du = \lim_{n\to\infty} \left(\int_\Omega u_n \operatorname{div}(z)\, dx + \int_\Omega z \cdot \nabla u_n\, dx \right)$$

$$= \lim_{n\to\infty} \int_{\partial\Omega} [z\cdot\nu] u_n\, d\mathcal{H}^{N-1} = \int_{\partial\Omega} [z\cdot\nu] u\, d\mathcal{H}^{N-1}. \qquad \blacksquare$$

The next theorem will be useful in the study of the Neumann problem.

Theorem 4.6. *Assume that ϕ is a metric integrand. Let $z \in \mathcal{A}_\infty(\Omega)$ be such that $[z\cdot\nu] = 0$ on $\partial\Omega$, $u \in BV_\phi(\Omega)$. Then*

$$\int_\Omega u \operatorname{div}(z)\, dx + \int_\Omega z \cdot Du = 0. \tag{4.9}$$

Proof. By Proposition 3.5, there exists a sequence $u_n \in W^{1,1}(\Omega)$ converging to u in $L^1(\Omega)$ such that $\int_\Omega \phi(x, \nabla u_n) \to \int_\Omega |Du|_\phi$. Then, by Lemma A.2, we have

$$\int_\Omega u \operatorname{div}(z)\, dx + \int_\Omega z \cdot Du = \lim_{n\to\infty} \left(\int_\Omega u_n \operatorname{div}(z)\, dx + \int_\Omega z \cdot \nabla u_n\, dx \right)$$

$$= \lim_{n\to\infty} \int_{\partial\Omega} [z\cdot\nu] u_n\, d\mathcal{H}^{N-1} = 0. \qquad \blacksquare$$

5. The subdifferential of the ϕ-total variation. In this section *we assume that $\phi : \Omega \times \mathbb{R}^N \to [0,\infty)$ is a continuous and coercive metric integrand in Ω.* Notice that in this case $BV_\phi(\Omega) = BV(\Omega)$. Let us define the functional

$$\psi_\phi(u) := \begin{cases} \displaystyle\int_\Omega \phi(x, \nabla u) & \text{if } u \in L^2(\Omega) \cap W^{1,1}(\Omega) \text{ and } u = 0 \text{ on } \partial\Omega, \\ +\infty & \text{otherwise.} \end{cases} \tag{5.1}$$

According to [49, 5], the functional $\Psi_\phi : L^2(\Omega) \to (-\infty, +\infty]$ defined by

$$\Psi_\phi(u) := \begin{cases} \displaystyle\int_\Omega |Du|_\phi + \int_{\partial\Omega} \phi(x, \nu^\Omega)|u|\, d\mathcal{H}^{N-1} & \text{if } u \in L^2(\Omega) \cap BV(\Omega), \\ +\infty & \text{otherwise} \end{cases} \tag{5.2}$$

is the lower semicontinuous relaxation of ψ_ϕ. Moreover, $\Psi_\phi(u)$ is lower semicontinuous with respect to convergence in $L^1(\Omega)$ [49]. Since Ψ_ϕ is convex and lower semicontinuous in $L^2(\Omega)$, we have that $\partial\Psi_\phi$ is a maximal monotone operator in $L^2(\Omega)$ (see [14]). The following theorem gives the characterization of $\partial\Psi_\phi$ (see [49] for a proof).

Theorem 5.1. *The following conditions are equivalent:*

(i) $v \in \partial\Psi_\phi(u)$.

(ii) *$u, v \in L^2(\Omega)$, $u \in BV(\Omega)$, and there exists $\sigma \in X_2(\Omega)$ with $\phi^0(x, \sigma(x)) \le 1$ a.e. in Ω, $v = -\mathrm{div}(\sigma)$ in $\mathcal{D}'(\Omega)$ such that $\sigma(x) \in \partial_\xi \phi(x, \nabla u(x))$ a.e. in Ω, $\sigma \cdot Du = |Du|_\phi$, and $[\sigma \cdot \nu^\Omega] \in \mathrm{sign}(-u)\phi(x, \nu^\Omega(x))$ $\mathcal{H}^{N-1}$-a.e. in $\partial\Omega$.*

(iii) *$u, v \in L^2(\Omega)$, $u \in BV(\Omega)$, and there exists $\sigma \in X_2(\Omega)$ with $\phi^0(x, \sigma(x)) \le 1$ a.e. in Ω, $v = -\mathrm{div}(\sigma)$ in $\mathcal{D}'(\Omega)$ such that*

$$\int_\Omega (w-u)v \le \int_\Omega \sigma \cdot Dw - \int_\Omega |Du|_\phi - \int_{\partial\Omega} [\sigma \cdot \nu^\Omega] w - \int_{\partial\Omega} \phi(x, \nu^\Omega(x))|u|$$

for all $w \in BV(\Omega) \cap L^2(\Omega)$.

(iv) *$u, v \in L^2(\Omega)$, $u \in BV(\Omega)$, and there exists $\sigma \in X_2(\Omega)$ with $\phi^0(x, \sigma(x)) \le 1$ a.e. in Ω, $v = -\mathrm{div}(\sigma)$ in $\mathcal{D}'(\Omega)$ such that*

$$\int_\Omega (w-u)v \le \int_\Omega \sigma \cdot Dw - \int_\Omega |Du|_\phi + \int_{\partial\Omega} \phi(x, \nu^\Omega(x))|w| - \int_{\partial\Omega} \phi(x, \nu^\Omega(x))|u|$$

for all $w \in BV(\Omega) \cap L^2(\Omega)$.

When we used in the previous statement the expression a.e. in Ω we mean a.e. with respect to the Lebesgue measure in Ω. The identity $\sigma \cdot Du = |Du|_\phi$ means that both Radon measures coincide.

From now on we shall write $v = -\mathrm{div}\, \partial_\xi \phi(x, \nabla u)$ instead of $v \in \partial \Psi_\phi(u)$.

6. The maximal ϕ-Cheeger set inside Ω.

Let Ω, Q be open bounded sets in $\mathbb{R}^N$ with Lipschitz boundary such that $\Omega \subset\subset Q$. Let $\phi : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a metric integrand with an extension to $Q \times \mathbb{R}^N$ such that *ϕ is continuous and coercive in a neighborhood of $Q \setminus \Omega$. For the rest of the paper we assume that this property holds.* Let $h \in L^\infty(\Omega)$, $h(x) > 0$ a.e. in Ω, such that

$$\int_\Omega \frac{1}{h(x)}\, dx < \infty. \tag{6.1}$$

We denote by $L^2(\Omega, hdx)$ the set of measurable functions $u : \Omega \to \mathbb{R}$ such that $\int_\Omega u^2\, h\, dx < \infty$. From (6.1) we have that $L^2(\Omega) \subseteq L^2(\Omega, hdx)$. For $f \in L^2(\Omega, hdx)$, $\lambda > 0$, let us consider the energy functional

$$\mathcal{E}_{\phi,h,\lambda}(u) := \int_\Omega |Du|_\phi + \frac{\lambda}{2} \int_\Omega (u-f)^2 h\, dx + \int_{\partial\Omega} \phi(x, \nu^\Omega)|u|\, d\mathcal{H}^{N-1}. \tag{6.2}$$

Although for ϕ-Cheeger sets we need only the case $f = 1$, the general case where $f \ne 1$ is of interest in section 11, where we discuss the application to anisotropic diffusion.

Let us consider the partial differential equation formally related to (6.2):

$$hu - \lambda^{-1} \mathrm{div}\,(\partial_\xi \phi(x, Du)) = hf \tag{6.3}$$

with Dirichlet boundary conditions. Notice that this is a symbolic notation. There is also a slight abuse of notation in writing (6.3) as an equality. Since the subdifferential of the ϕ-total variation is multivalued, (6.3) would be better written as $hf \in hu - \lambda^{-1}\mathrm{div}\,(\partial_\xi \phi(x, \nabla u))$. In spite of this we will write the equation as (6.3), understanding that the equality holds for an element of the subdifferential.

Definition 6.1. *Let $f \in L^\infty(\Omega)$. We say that $u \in L^2(\Omega, h\, dx)$ is a solution of (6.3) if $u \in BV_\phi(\Omega) \cap L^\infty(\Omega)$, and there is a vector field $\sigma \in \mathcal{A}_\infty(\Omega)$ such that*

(i) $hu - \lambda^{-1}\mathrm{div}\,(\sigma) = hf$ *in* $\mathcal{D}'(\Omega)$,
(ii) $\phi^0(x,\sigma(x)) \le 1$ *a.e.*,
(iii) $\int_\Omega \sigma \cdot Du = \int_\Omega |Du|_\phi$,
(iv) $[\sigma \cdot \nu^\Omega] \in \mathrm{sign}(-u)\phi(x,\nu^\Omega(x))$, $\mathcal{H}^{N-1}$*-a.e.* $x \in \partial\Omega$.

We could have given a more general definition, but the present case is sufficient for our purposes.

Theorem 6.2. (i) *Let* $f \in L^2(\Omega, hdx)$. *Then there is a unique solution of the problem*

$$(Q)_\lambda: \quad \min_{u \in BV_\phi(\Omega)\cap L^2(\Omega,h\,dx)} \quad \mathcal{E}_{\phi,h,\lambda}(u). \tag{6.4}$$

(ii) *Assume that* $f \in L^\infty(\Omega)$. *Then there is a unique solution* $u \in L^2(\Omega, h\,dx)$ *of* (6.3). *Moreover, the solution* u *belongs to* $L^\infty(\Omega)$, *and it minimizes* (6.4).

Proof. (i) Let u_n be a minimizing sequence for (6.4). Then u_n is bounded in $L^2(\Omega, h\,dx)$. Assume that $u_n \to u$ weakly in $L^2(\Omega, h\,dx)$. Observe that $u \in L^1(\Omega)$ since

$$\int_\Omega |u|\,dx \le \left(\int_\Omega u^2\,h\,dx\right)^{1/2}\left(\int_\Omega \frac{1}{h}\,dx\right)^{1/2} \tag{6.5}$$

and $u_n \to u$ weakly in $L^1(\Omega)$. Indeed if $\varphi \in L^\infty(\Omega)$, then $\frac{\varphi}{h} \in L^2(\Omega, h\,dx)$ and

$$\int_\Omega (u_n - u)\varphi\,dx = \int_\Omega (u_n - u)\frac{\varphi}{h}\,h\,dx \to 0.$$

For any given function $v \in L^1(\Omega)$, let $\tilde v$ denote its extension by 0 in $\mathbb{R}^N$. Since $v \to \int_\Omega |D\tilde v|_\phi$ is convex and lower semicontinuous with respect to the convergence in $L^1(\Omega)$, it is also lower semicontinuous with respect to the weak convergence in $L^1(\Omega)$. Hence by Lemma 3.9 we have

$$\begin{aligned}\int_\Omega |Du|_\phi + \int_{\partial\Omega}\phi(x,\nu^\Omega)|u| &= \int_{\mathbb{R}^N}|D\tilde u|_\phi \le \liminf_n \int_{\mathbb{R}^N}|D\tilde u_n|_\phi\\ &= \liminf_n \int_\Omega |Du_n|_\phi + \int_{\partial\Omega}\phi(x,\nu^\Omega)|u_n|.\end{aligned}$$

Then we also have

$$\mathcal{E}_{\phi,h,\lambda}(u) \le \liminf_n \mathcal{E}_{\phi,h,\lambda}(u_n),$$

and $u \in BV_\phi(\Omega) \cap L^2(\Omega, h\,dx)$ is a minimum of $\mathcal{E}_{\phi,h,\lambda}(u)$. Since the functional is strictly convex, the solution is unique.

(ii) We divide the proof into three steps.

Step 1. *Existence and uniqueness of solutions of an approximating problem.* Let $h_n = h + \frac{1}{n}$, $\phi_n(x,\xi) = \phi(x,\xi) + \frac{1}{n}\Xi(\xi)$, where $\Xi(\xi) = |\xi|$, $x \in \Omega$, $\xi \in \mathbb{R}^N$. As in (i) there is a unique minimizer u_n of $\mathcal{E}_{\phi_n,h_n,\lambda}(u)$ which is in $BV_{\phi_n}(\Omega) \cap L^2(\Omega, h_n\,dx)$. Since ϕ_n is coercive and $h_n \ge \frac{1}{n}$, it follows that $u_n \in BV(\Omega) \cap L^2(\Omega)$. As a consequence we have that

$$\lambda(f - u_n)h_n \in \partial\Psi_{\phi_n}(u_n), \tag{6.6}$$

the subdifferential $\partial\Psi_{\phi_n}(u_n)$ being taken in $L^2(\Omega)$. Now, since ϕ_n is continuous and coercive, by the characterization of the subdifferential $\partial\Psi_{\phi_n}(u_n)$ given in Theorem 5.1, u_n satisfies the equation

$$h_n u_n - \lambda^{-1}\mathrm{div}\,(\partial\phi_n(x,\nabla u_n)) = h_n f \tag{6.7}$$

in the sense of Definition 6.1. That is, there exists $z_n \in \partial\phi_n(x,\nabla u_n)$, $z_n \in X_2(\Omega)$, such that

$$h_n u_n - \lambda^{-1}\mathrm{div}\, z_n = h_n f \qquad \text{in } \mathcal{D}'(\Omega), \tag{6.8}$$

$$\int_\Omega z_n \cdot Du_n = \int_\Omega |Du_n|_{\phi_n}, \tag{6.9}$$

$$[z_n \cdot \nu^\Omega] \in \mathrm{sign}(-u_n)\phi_n(x,\nu^\Omega(x)), \qquad \mathcal{H}^{N-1}\text{-a.e. } x \in \partial\Omega. \tag{6.10}$$

Conversely, using again Theorem 5.1, if $u_n \in L^2(\Omega)$ is a solution of (6.7), then it is also a solution of (6.6). Now, since Ψ_{ϕ_n} is a maximal monotone operator in $L^2(\Omega)$, the uniqueness of solutions of (6.7) follows immediately by standard results [14].

Since $\partial\phi_n = \partial\phi + \frac{1}{n}\partial\Xi$, we may write $z_n = \sigma_n + \frac{1}{n}\eta_n$, where $\sigma_n \in \partial\phi(x,\nabla u_n)$ and $\eta_n \in \partial\Xi(\nabla u_n)$.

Step 2. *Basic estimates and passage to the limit.* Assume that $a \le f \le b$. Let us prove that $a \le u_n \le b$, $a, b \in \mathbb{R}$. First we observe that, multiplying (6.7) by u_n and integrating by parts, we obtain

$$\int_\Omega u_n^2\, h_n\, dx + \lambda^{-1}\int_\Omega |Du_n|_{\phi_n} + \lambda^{-1}\int_{\partial\Omega} \phi_n(x,\nu^\Omega)|u_n| = \int_\Omega f\, h_n\, u_n\, dx, \tag{6.11}$$

which implies that u_n is uniformly bounded in $BV_\phi(\Omega)$. To prove that $u_n \le b$, we multiply $h_n(u_n - b) - \lambda^{-1}\mathrm{div}\, z_n = h_n(f-b)$ by $(u_n-b)^+$, and, integrating by parts, we obtain

$$\begin{aligned}\int_\Omega ((u_n-b)^+)^2\, h_n\, dx + \lambda^{-1}\int_\Omega z_n \cdot D(u_n-b)^+ + \lambda^{-1}\int_{\partial\Omega} \phi_n(x,\nu^\Omega)(u_n-b)^+\, d\mathcal{H}^{N-1} \\ = \int_\Omega (f-b)(u_n-b)^+ h_n\, dx \le 0.\end{aligned}$$

Using Lemma 4.2 we have that $\int_\Omega z_n \cdot D(u_n-b)^+ \ge 0$. Since the third term on the left-hand side is also ≥ 0, we have that $(u_n-b)^+ = 0$; i.e., $u_n \le b$ a.e. In a similar way we prove that $u_n \ge a$ a.e. Modulo a subsequence, we may assume that $u_n \to u$ weakly in $L^2(\Omega)$ for some $u \in BV_\phi(\Omega) \cap L^\infty(\Omega)$. Since $h_n \to h$ uniformly in Ω, it also holds that

$$u_n h_n \to uh \quad \text{weakly in } L^2(\Omega) \text{ as } n \to \infty. \tag{6.12}$$

Finally, since z_n, η_n are bounded in $L^\infty(\Omega)$, by extracting a further subsequence we may assume that $z_n, \sigma_n \to \sigma$ weakly* in $L^\infty(\Omega)$. Now, since $\phi^0(x,\sigma_n(x)) \le 1$ a.e., and this condition is stable under weak* convergence in $L^\infty(\Omega)$, we have $\phi^0(x,\sigma(x)) \le 1$ a.e. Now, since $\mathrm{div}\, z_n$ is bounded in $L^\infty(\Omega)$, we have that, by the Banach–Alaoglu theorem,

$$\mathrm{div}\, z_n \to \mathrm{div}\,\sigma \qquad \text{weakly in } L^2(\Omega)$$

and $\sigma \in \mathcal{A}_\infty(\Omega)$. Letting $n \to \infty$ in (6.8), we obtain that

$$hu - \lambda^{-1}\mathrm{div}\,\sigma = hf \qquad \text{in } \mathcal{D}'(\Omega). \tag{6.13}$$

Step 3. *Final step.* Let us prove that

$$\int_\Omega \sigma \cdot Du = \int_\Omega |Du|_\phi, \tag{6.14}$$

$$[\sigma \cdot \nu^\Omega] \in \mathrm{sign}(-u)\phi(x, \nu^\Omega) \quad \text{a.e. on } \partial\Omega. \tag{6.15}$$

Let $\varphi \in C^1(\Omega) \cap C(\overline{\Omega})$. Since $|[z_n \cdot \nu^\Omega]| \leq \phi_n(x, \nu^\Omega(x))$ a.e. in $\partial\Omega$, we may assume that $[z_n \cdot \nu^\Omega] \to \beta(x)$ weakly* in $L^\infty(\partial\Omega)$, and, letting $n \to \infty$ in

$$\int_\Omega z_n \cdot \nabla\varphi\, dx + \int_{\partial\Omega} [z_n \cdot \nu^\Omega]\varphi\, d\mathcal{H}^{N-1} = -\int_\Omega \mathrm{div} z_n\, \varphi\, dx,$$

we obtain

$$\int_\Omega \sigma \cdot \nabla\varphi\, dx + \int_{\partial\Omega} \beta\varphi\, d\mathcal{H}^{N-1} = -\int_\Omega \mathrm{div}\sigma\, \varphi\, dx = \int_\Omega \sigma \cdot \nabla\varphi\, dx + \int_{\partial\Omega} [\sigma \cdot \nu^\Omega]\varphi\, d\mathcal{H}^{N-1}.$$

Hence

$$\int_{\partial\Omega} \beta\varphi\, d\mathcal{H}^{N-1} = \int_{\partial\Omega} [\sigma \cdot \nu^\Omega]\varphi\, d\mathcal{H}^{N-1}$$

holds for any $\varphi \in C^1(\Omega) \cap C(\overline{\Omega})$, and we obtain that $\beta = [\sigma \cdot \nu^\Omega]$. In particular

$$|[\sigma \cdot \nu^\Omega]| \leq \phi(x, \nu^\Omega(x)) \quad \text{a.e. in } \partial\Omega. \tag{6.16}$$

Now, using (6.12) in the fifth line of the following computations and using Corollary A.1 (see Appendix A.2) together with $\|\phi^0(x, \sigma(x))\|_\infty \leq 1$ and (6.16) in the last inequality below, we have

$$\begin{aligned}
&\int_\Omega u^2\, h\, dx + \lambda^{-1} \int_\Omega |Du|_\phi + \lambda^{-1} \int_{\partial\Omega} \phi(x, \nu^\Omega)|u|\, d\mathcal{H}^{N-1} \\
&\leq \liminf_n \int_\Omega u_n^2\, h_n\, dx + \liminf_n \left\{ \lambda^{-1} \int_\Omega |Du_n|_\phi + \lambda^{-1} \int_{\partial\Omega} \phi(x, \nu^\Omega)|u_n|\, d\mathcal{H}^{N-1} \right\} \\
&\leq \liminf_n \int_\Omega u_n^2\, h_n\, dx + \liminf_n \left\{ \lambda^{-1} \int_\Omega |Du_n|_{\phi_n} + \lambda^{-1} \int_{\partial\Omega} \phi_n(x, \nu^\Omega)|u_n|\, d\mathcal{H}^{N-1} \right\} \\
&\leq \liminf_n \left\{ \int_\Omega u_n^2\, h_n\, dx + \lambda^{-1} \int_\Omega |Du_n|_{\phi_n} + \lambda^{-1} \int_{\partial\Omega} \phi_n(x, \nu^\Omega)|u_n|\, d\mathcal{H}^{N-1} \right\} \\
&= \liminf_n \int_\Omega f u_n\, h_n\, dx = \int_\Omega f u\, h\, dx = \int_\Omega u^2\, h\, dx - \lambda^{-1} \int_\Omega u \mathrm{div}\, \sigma\, dx \\
&= \int_\Omega u^2\, h\, dx + \lambda^{-1} \int_\Omega \sigma \cdot Du - \lambda^{-1} \int_{\partial\Omega} [\sigma \cdot \nu^\Omega] u\, d\mathcal{H}^{N-1} \\
&\leq \int_\Omega u^2\, h\, dx + \lambda^{-1} \int_\Omega |Du|_\phi + \lambda^{-1} \int_{\partial\Omega} \phi(x, \nu^\Omega)|u|\, d\mathcal{H}^{N-1}.
\end{aligned}$$

In particular, we obtain (6.14) and

$$-\int_{\partial\Omega}[\sigma\cdot\nu^{\Omega}]u\,d\mathcal{H}^{N-1}=\int_{\partial\Omega}\phi(x,\nu^{\Omega})|u|\,d\mathcal{H}^{N-1}.$$

The last identity implies that $-[\sigma\cdot\nu^{\Omega}]u=\phi(x,\nu^{\Omega})|u|$; hence (6.15) follows. Thus, conditions (i)–(iv) of Definition 6.1 are satisfied and u is a solution of (6.3).

The proof of uniqueness follows in a standard way (see [8, Chapter 2]). Finally, since the energy is convex, the solution of (6.3) is a minimizer of (6.4). ∎

Proposition 6.3. *Let $u\in BV_{\phi}(\Omega)\cap L^2(\Omega,h\,dx)$ be the solution of the variational problem (6.4) with $f=1$. Then $0\le u\le 1$. Let $E_s:=\{u\ge s\}$, $s\in(0,1]$. Then for any $s\in(0,1]$ we have*

$$P_{\phi}(E_s)-\lambda(1-s)|E_s|_h\le P_{\phi}(F)-\lambda(1-s)|F|_h \tag{6.17}$$

for any $F\subseteq\Omega$.

We denote $|F|_h=\int_F h(x)\,dx$ for any measurable subset $F\subseteq\Omega$.

Proof. Recall that u satisfies the following PDE:

$$hu-\lambda^{-1}\operatorname{div}\sigma=h\quad\text{in }\Omega, \tag{6.18}$$

where $\sigma(x)\in\partial_{\xi}\phi(x,\nabla u(x))$ a.e. As in Step 2 of the proof of Theorem 6.2(ii) we deduce that $0\le u\le 1$.

Let us prove that for almost any $s\in(0,\infty)$ we have

$$P_{\phi}(E_s)=\int_{\Omega}\sigma\cdot D\chi_{E_s}+\int_{\partial\Omega}\phi(x,\nu^{\Omega}(x))\chi_{E_s}(x)\,d\mathcal{H}^{N-1}.$$

Indeed, multiplying (6.18) by u and integrating by parts we obtain

$$\int_{\Omega}\sigma\cdot Du+\int_{\partial\Omega}\phi(x,\nu^{\Omega})|u|\,d\mathcal{H}^{N-1}=\lambda\int_{\Omega}(1-u)uh\,dx. \tag{6.19}$$

Now, multiplying (6.18) by χ_{E_s} and integrating by parts we obtain

$$\int_{\Omega}\sigma\cdot D\chi_{E_s}-\int_{\partial\Omega}[\sigma\cdot\nu^{\Omega}]\chi_{E_s}\,d\mathcal{H}^{N-1}=\lambda\int_{\Omega}(1-u)h\chi_{E_s}\,dx. \tag{6.20}$$

Notice that this relation proves that $\int_{\Omega}\sigma\cdot D\chi_{E_s}-\int_{\partial\Omega}[\sigma\cdot\nu^{\Omega}]\chi_{E_s}\,d\mathcal{H}^{N-1}$ is a measurable function of s. Integrating (6.20) with respect to s we obtain

$$\begin{aligned}\int_0^{\infty}\int_{\Omega}\sigma\cdot D\chi_{E_s}\,ds-\int_0^{\infty}\int_{\partial\Omega}[\sigma\cdot\nu^{\Omega}]\chi_{E_s}\,d\mathcal{H}^{N-1}\,ds&=\lambda\int_0^{\infty}\int_{\Omega}(1-u)h\chi_{E_s}\,dx\,ds\\&=\lambda\int_{\Omega}(1-u)h\int_0^{\infty}\chi_{E_s}\,dsdx=\lambda\int_{\Omega}(1-u)uh\,dx\\&=\int_{\Omega}\sigma\cdot Du+\int_{\partial\Omega}\phi(x,\nu^{\Omega})|u|\,d\mathcal{H}^{N-1}\\&=\int_{\Omega}|Du|_{\phi}+\int_{\partial\Omega}\phi(x,\nu^{\Omega})|u|\,d\mathcal{H}^{N-1}.\end{aligned}$$

Now, using Lemma 3.9 and the coarea formula (3.6), we have

$$\int_\Omega |Du|_\phi + \int_{\partial\Omega} \phi(x,\nu^\Omega)|u|\, d\mathcal{H}^{N-1} = \int_{\mathbb{R}^N} |D\tilde{u}|_\phi = \int_0^\infty P_\phi(\{\tilde{u} > s\})\, ds$$
$$= \int_0^\infty P_\phi(\{u > s\})\, ds,$$

where $\tilde{u}$ denotes the extension of u by zero outside Ω. Hence

$$\int_0^\infty \int_\Omega \sigma \cdot D\chi_{E_s}\, ds - \int_0^\infty \int_{\partial\Omega} [\sigma\cdot\nu^\Omega]\chi_{E_s}\, d\mathcal{H}^{N-1}\, ds = \int_0^\infty P_\phi(\{u > s\})\, ds. \tag{6.21}$$

Let $s > 0$. Now, using Proposition 4.1 and the fact that $[\sigma\cdot\nu^\Omega] \in \operatorname{sign}(-u)\phi(x,\nu^\Omega(x))$, we have

$$\left| \int_\Omega \sigma\cdot D\chi_{E_s} - \int_{\partial\Omega} [\sigma\cdot\nu^\Omega]\chi_{E_s}\, d\mathcal{H}^{N-1} \right| \le \int_\Omega |D\chi_{E_s}|_\phi + \int_{\partial\Omega} \phi(x,\nu^\Omega(x))\chi_{E_s}\, d\mathcal{H}^{N-1},$$

and using again Lemma 3.9 and the coarea formula, we may continue the equalities and obtain

$$\int_\Omega |D\chi_{E_s}|_\phi + \int_{\partial\Omega} \phi(x,\nu^\Omega(x))\chi_{E_s}\, d\mathcal{H}^{N-1} = \int_\Omega |D\tilde{\chi}_{E_s}|_\phi = \int_\Omega |D\chi_{E_s}|_\phi = P_\phi(E_s),$$

where $\tilde{\chi}_{E_s}$ is the extension of χ_{E_s} by zero outside Ω. Combining this inequality with (6.21), we obtain

$$\int_\Omega \sigma\cdot D\chi_{E_s} - \int_{\partial\Omega} [\sigma\cdot\nu^\Omega]\chi_{E_s}\, d\mathcal{H}^{N-1} = P_\phi(E_s) \qquad \text{a.e. } s > 0. \tag{6.22}$$

Let $F \subseteq \Omega$ be a set of ϕ-finite perimeter. For $s \in (0,1]$, we have

$$-\int_\Omega \operatorname{div}\sigma\,(\chi_F - \chi_{E_s})\, dx = \int_\Omega (\sigma, D\chi_F) - \int_\Omega \sigma\cdot D\chi_{E_s} - \int_{\partial\Omega} [\sigma\cdot\nu^\Omega](\chi_F - \chi_{E_s})\, d\mathcal{H}^{N-1}$$
$$= \int_\Omega (\sigma, D\chi_F) - \int_{\partial\Omega} [\sigma\cdot\nu^\Omega]\chi_F - P_\phi(E_s) \le P_\phi(F) - P_\phi(E_s),$$

and we deduce

$$P_\phi(F) - P_\phi(E_s) \ge \lambda \int_\Omega (1-u)h(\chi_F - \chi_{E_s}) = \lambda \int_\Omega ((1-s) + (s-u))h(\chi_F - \chi_{E_s}).$$

Since $(s-u)(\chi_F - \chi_{E_s}) \ge 0$, we have

$$P_\phi(F) - P_\phi(E_s) \ge \lambda \int_\Omega (1-s)h(\chi_F - \chi_{E_s}) = \lambda(1-s)(|F|_h - |E_s|_h).$$

Since all sets E_s are contained in Ω, the perimeter is lower semicontinuous, and the area is continuous for increasing or decreasing families of sets contained in Ω, we deduce that (6.17) holds for any $s \in (0,1]$. ■

Remark 3. Let us observe that (6.22) holds for any $s > 0$. Indeed, by lower semicontinuity we have that $P_\phi(E_s) < \infty$ for all $s > 0$. If $s > 0$, we may approximate it by s_n such that (6.22) holds for s_n. Since

$$P_\phi(E_{s_n}) = \int_\Omega \sigma \cdot D\chi_{E_{s_n}} - \int_{\partial\Omega} [\sigma \cdot \nu^\Omega] \chi_{E_{s_n}} \, d\mathcal{H}^{N-1} = - \int_\Omega \operatorname{div} \sigma \, \chi_{E_{s_n}},$$

we have $P_\phi(E_s) \le \liminf_n P_\phi(E_{s_n}) = -\int_\Omega \operatorname{div} \sigma \, \chi_{E_s} = \int_\Omega \sigma \cdot D\chi_{E_s} - \int_{\partial\Omega} [\sigma \cdot \nu^\Omega] \chi_{E_s} \, d\mathcal{H}^{N-1}$ $\le P_\phi(E_s)$. The identity (6.22) holds for any $s > 0$.

Lemma 6.4. *Let u_λ be the solution of $(Q)_\lambda$. If*

$$\frac{1}{\lambda} < \|h\chi_\Omega\|_* := \sup \left\{ \left| \int_\Omega hu \, dx \right| : u \in L^2(\Omega) \cap BV_\phi(\Omega), \right.$$
$$\left. \int_\Omega |Du|_\phi + \int_{\partial\Omega} \phi(x, \nu^\Omega) |u| \, d\mathcal{H}^{N-1} \le 1 \right\},$$

then $u_\lambda \neq 0$.

Proof. Notice that u_λ is characterized as the solution of (6.3). If $u_\lambda = 0$, then there exists a vector field $\sigma \in X_\infty(\Omega)$ with $\phi^0(x, \sigma(x)) \le 1$ a.e., $|[\sigma \cdot \nu^\Omega]| \le \phi(x, \nu^\Omega(x))$ $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$, and $-\lambda^{-1} \operatorname{div} \sigma = h\chi_\Omega$ in $\mathcal{D}'(\Omega)$. Multiplying by $u \in BV_\phi(\Omega)$ and integrating by parts, we obtain that $\|h\chi_\Omega\|_* \le \frac{1}{\lambda}$. ■

If $\phi : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a metric integrand and ϕ_1, ϕ_2 are two extensions of ϕ to open sets Ω', Ω'', respectively, such that $\Omega \subset\subset \Omega', \Omega''$, and they coincide in $\Omega' \cap \Omega''$, then

$$P_{\phi_1}(E, \Omega') = P_{\phi_2}(E, \Omega'') = P_{\phi_i}(E, \Omega' \cap \Omega''), \qquad i = 1, 2, \ E \subseteq \Omega.$$

In particular, if ϕ has an extension to $\mathbb{R}^N$, denoted again by ϕ, then the above perimeters are also equal to $P_\phi(E)$. Thus, if ϕ has an extension to an open set Q containing $\overline{\Omega}$, we shall denote $P_\phi(E)$ instead of $P_\phi(E, Q)$ for any set $E \subseteq \Omega$ with finite ϕ-perimeter. Notice that if ϕ is continuous and coercive in a neighborhood of $Q \setminus \Omega$, then

$$P_\phi(E) = P_\phi(E, \Omega) + \int_{\partial\Omega} \phi(x, \nu^\Omega) \, \chi_E(x) \, d\mathcal{H}^{N-1}, \qquad E \subseteq \Omega.$$

Notice that in this case, $P_\phi(E)$ depends only on $\phi : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$.

Lemma 6.5. *Let Ω, Q be open bounded sets with Lipschitz boundary such that $\Omega \subset\subset Q$. Let $\phi : Q \times \mathbb{R}^N \to \mathbb{R}$ be a metric integrand which is continuous and coercive in a neighborhood of $Q \setminus \Omega$. If E, F are two sets of finite ϕ-perimeter, then*

$$P_\phi(E \cup F) + P_\phi(E \cap F) \le P_\phi(E) + P_\phi(F). \tag{6.23}$$

Using the results of section 3.2, the proof is exactly the same as in [35] for the case of the Euclidean perimeter, and we omit the details.

The following lemma can be proved as in [4], and we also omit the details.

Lemma 6.6. *For any $\lambda > 0$, let us consider the problem*

$$(P)_\lambda : \quad \min_{F \subseteq \Omega} P_\phi(F) - \lambda |F|_h. \tag{6.24}$$

Then the following hold:

(i) *Let* C_λ, C_μ *be minimizers of* $(P)_\lambda$, *and* $(P)_\mu$, *respectively. If* $\lambda < \mu$, *then* $C_\lambda \subseteq C_\mu$.
(ii) *Let* $\mu > \lambda$. *Assume that* Ω *is a solution of* $(P)_\lambda$. *Then* Ω *is a solution of* $(P)_\mu$.
(iii) *Let* $\lambda_n \uparrow \lambda$. *Then* $C_\lambda^+ := \cup_n C_{\lambda_n}$ *is a minimizer of* $(P)_\lambda$. *Moreover,* $P_\phi(C_{\lambda_n}) \to P_\phi(C_\lambda^+)$. *Similarly, if* $\lambda_n \downarrow \lambda$, *then* $C_\lambda^- := \cap_n C_{\lambda_n}$ *is a minimizer of* $(P)_\lambda$, *and* $P_\phi(C_{\lambda_n}) \to P_\phi(C_\lambda^-)$.
(iv) *If* $\lambda \to \infty$ *and* C_λ *is a minimizer of* $(P)_\lambda$, *then* $C_\lambda \to \Omega$.

Remark 4. In Proposition 6.3 we have proved that for any $s \in (0,1]$, the level set $\{u_\lambda \geq s\}$ is a minimizer of $(P)_{\lambda(1-s)}$. Moreover, by Lemma 6.6, the sets $\{u_\lambda \geq s\}^+ := \cup_{\epsilon>0}\{u_\lambda \geq s+\epsilon\}$, $s \in [0,1)$, and $\{u_\lambda \geq s\}^- := \cap_{\epsilon>0}\{u_\lambda \geq s-\epsilon\}$, $s \in (0,1]$, are also minimizers of $(P)_{\lambda(1-s)}$ (obviously $\{u_\lambda \geq 1\}^+ = \emptyset$ is also a minimizer of $(P)_0$). Notice that, except on countably many values of s, they coincide with $[u_\lambda \geq s]$.

As a consequence of Lemma 6.6.(ii), we obtain the following corollary.

Corollary 6.7. *Then for almost any* λ, $(P)_\lambda$ *has a unique solution.*

From Proposition 6.7 and Lemma 6.6(iii) we deduce the following consequence.

Proposition 6.8. *Let* $\alpha, \beta > \frac{1}{\|h\chi_\Omega\|_*}$. *Then* $\alpha(1-\|u_\alpha\|_\infty) = \beta(1-\|u_\beta\|_\infty)$.

Proof. Assume that these two numbers are not equal. Without loss of generality, we may assume that

$$\alpha(1-\|u_\alpha\|_\infty) < \beta(1-\|u_\beta\|_\infty).$$

Let us take λ such that the solution of $(P)_\lambda$ is unique and $\alpha(1-\|u_\alpha\|_\infty) < \lambda < \beta(1-\|u_\beta\|_\infty)$. Let us write $\lambda = \alpha(1-s) = \beta(1-t)$ for some values $s < \|u_\alpha\|_\infty$ and $t > \|u_\beta\|_\infty$. Since $\{u_\beta \geq t\} = \emptyset$, and the solution of $(P)_\lambda$ is unique, being $\{u_\alpha \geq s\}$ a solution of $(P)_\lambda$, we deduce that $\{u_\alpha \geq s\} = \emptyset$, a contradiction. This proves our proposition. ∎

Let λ^* be the unique value of $\alpha(1-\|u_\alpha\|_\infty)$ determined by the above proposition.

Proposition 6.9. *Let* $\alpha, \beta > \frac{1}{\|h\chi_\Omega\|_*}$. *Then* $\{u_\alpha \geq \|u_\alpha\|_\infty\} = \{u_\beta \geq \|u_\beta\|_\infty\}$, *and*

$$\lambda^* = \frac{P(\{u_\alpha \geq \|u_\alpha\|_\infty\})}{|\{u_\alpha \geq \|u_\alpha\|_\infty\}|_h}. \tag{6.25}$$

The set $\{u_\alpha \geq \|u_\alpha\|_\infty\}$ *is the maximal* ϕ*-Cheeger set of* Ω.

Proof. Let $\delta_n \to 0+$ be such that $(P)_{\lambda^*+\delta_n}$ has a unique solution for each n. Since $\{u_\alpha \geq \|u_\alpha\|_\infty - \frac{\delta_n}{\alpha}\}$, $\{u_\beta \geq \|u_\beta\|_\infty - \frac{\delta_n}{\beta}\}$ are both solutions of $(P)_{\lambda^*+\delta_n}$, we have that

$$\left\{u_\alpha \geq \|u_\alpha\|_\infty - \frac{\delta_n}{\alpha}\right\} = \left\{u_\beta \geq \|u_\beta\|_\infty - \frac{\delta_n}{\beta}\right\}.$$

Since $\{u_\alpha \geq \|u_\alpha\|_\infty\} = \cap_n\{u_\alpha \geq \|u_\alpha\|_\infty - \frac{\delta_n}{\alpha}\}$ and $\{u_\beta \geq \|u_\beta\|_\infty\} = \cap_n\{u_\beta \geq \|u_\beta\|_\infty - \frac{\delta_n}{\beta}\}$, we deduce that $\{u_\alpha \geq \|u_\alpha\|_\infty\} = \{u_\beta \geq \|u_\beta\|_\infty\}$, and this set minimizes $(P)_{\lambda^*}$.

Now, since $\{u_\alpha \geq \|u_\alpha\|_\infty + \epsilon\} = \emptyset$ is a solution of $(P)_{\lambda^*-\lambda\epsilon}$, for all $\epsilon > 0$, by Lemma 6.6(iii), we have that $\emptyset$ is also a solution of $(P)_{\lambda^*}$. Then

$$P_\phi(\{u_\alpha \geq \|u_\alpha\|_\infty\}) - \lambda^*|\{u_\alpha \geq \|u_\alpha\|_\infty\}|_h = P_\phi(\emptyset) - \lambda^*|\emptyset|_h = 0,$$

and (6.25) follows. Since $\{u_\alpha \geq \|u_\alpha\|_\infty\}$ is a minimizer of $(P)_{\lambda^*}$, we deduce that

$$0 = P_\phi(\{u_\alpha \geq \|u_\alpha\|_\infty\}) - \lambda^*|\{u_\alpha \geq \|u_\alpha\|_\infty\}|_h \leq P_\phi(F) - \lambda^*|F|_h$$

for any set $F \subseteq \Omega$ of finite perimeter. Then

$$\frac{P_\phi(\{u_\alpha \geq \|u_\alpha\|_\infty\})}{|\{u_\alpha \geq \|u_\alpha\|_\infty\}|_h} \leq \frac{P_\phi(F)}{|F|_h}$$

for any set $F \subseteq \Omega$ of finite perimeter. Thus, the set $\{u_\alpha \geq \|u_\alpha\|_\infty\}$ is a ϕ-Cheeger set of Ω. Now, if C is any other ϕ-Cheeger set in Ω, then C is a solution of $(P)_{\lambda^*}$. Then $C \subseteq \{u_\alpha \geq \|u_\alpha\|_\infty - \frac{\delta_n}{\alpha}\}$ for all n. Then $C \subseteq \{u_\alpha \geq \|u_\alpha\|_\infty\}$. We conclude that $\{u_\alpha \geq \|u_\alpha\|_\infty\}$ is the maximal ϕ-Cheeger set of Ω. ∎

6.1. Local ϕ-Cheeger sets in Ω. *In this section we assume that ϕ is continuous and coercive in* Ω. Let $E \subseteq \mathbb{R}^N$ be a set of finite perimeter. We say that E is decomposable if there exists a partition (A, B) of E such that $P_\phi(E) = P_\phi(A) + P_\phi(B)$ and both $|A|$ and $|B|$ are strictly positive. We say that E is indecomposable if it is not decomposable; notice that the properties of being decomposable or indecomposable are invariant modulo Lebesgue null sets and that, according to our definition, any Lebesgue negligible set is indecomposable.

The following result was proved in [6] for the Euclidean perimeter. The proof easily extends to cover the case where ϕ is continuous and coercive in Ω, but it also follows from the Euclidean case since the assumptions on ϕ imply that

$$P_\phi(E) = \int_{\partial^* E} \phi(x, \nu^E(x))\, d\mathcal{H}^{N-1}$$

for any set $E \subseteq \mathbb{R}^N$ with finite perimeter.

Theorem 6.10. *Let E be a set with finite perimeter in $\mathbb{R}^N$. Then there exists a unique finite or countable family of pairwise disjoint indecomposable sets $\{E_i\}_{i\in I}$ such that $|E_i| > 0$ and $P_\phi(E) = \sum_i P_\phi(E_i)$. Moreover, the sets E_i are maximal indecomposable sets; i.e., any indecomposable set $F \subseteq E$ is contained modulo a Lebesgue null set in some set E_i.*

In view of the previous theorem, we call the sets E_i the ϕ-connected components of E.

Proposition 6.11. *Assume that ϕ is continuous and coercive in Ω. Let $u \in BV_\phi(\Omega) \cap L^2(\Omega, h\,dx)$ be the solution of (6.4). Let $t \in (0,1]$ and $E_t := \{u \geq t\}$. Let E_t' be a ϕ-connected component of E_t, and let $F_s = \{u \geq s\} \cap E_t'$, $s \geq t$. Then for any $s \in (0,1]$ we have*

$$P_\phi(F_s) - \lambda(1-s)|F_s|_h \leq P_\phi(F) - \lambda(1-s)|F|_h \tag{6.26}$$

for any $F \subseteq E_t'$. If $s = \max_{x\in E_t'} u(x)$, then F_s is a maximal ϕ-Cheeger set in E_t'.

The sets F_s will be called local ϕ-Cheeger sets.

Proof. Let $\{E_s^i\}_{i\in I}$ be the ϕ-connected components of E_s. Since $\chi_{E_s} = \sum_i \chi_{E_s^i}$, $|\chi_{E_s}| = \sum_i |\chi_{E_s^i}|$, $\sigma \cdot D\chi_{E_s} = \sum_i \sigma \cdot D\chi_{E_s^i}$, and

$$\int_\Omega \sigma \cdot D\chi_{E_s^i} - \int_{\partial\Omega} [\sigma \cdot \nu^\Omega]\chi_{E_s^i}\, d\mathcal{H}^{N-1} \leq P_\phi(E_s^i, \Omega) + \int_{\partial\Omega} \phi(x, \nu^\Omega)\chi_{E_s^i}\, d\mathcal{H}^{N-1} = P_\phi(E_s^i),$$

from the extension of (6.22) given in Remark 3 we have that

$$\int_\Omega \sigma \cdot D\chi_{E_s^i} - \int_{\partial\Omega} [\sigma \cdot \nu^\Omega]\chi_{E_s^i}\, d\mathcal{H}^{N-1} = P_\phi(E_s^i)$$

for any $i \in I$. Now, we can proceed as in the proof of Proposition 6.3 to get that (6.26) holds. The last assertion follows as in Propositions 6.8 and 6.9. ■

Recall that, when ϕ is coercive, by the isoperimetric inequality there is a constant $\alpha > 0$ (depending on the domain) such that any ϕ-Cheeger set has measure $\geq \alpha$. Moreover, the union and intersection of ϕ-Cheeger sets are ϕ-Cheeger [19]. In particular, there are minimal ϕ-Cheeger sets and there are finitely many of them [19].

7. Remarks on the Neumann problem. Let us now consider the energy functional

$$\mathcal{E}_\lambda(u) := \int_\Omega |Du|_\phi + \frac{\lambda}{2}\int_\Omega (u-f)^2 h\, dx, \quad f \in L^2(\Omega, hdx). \tag{7.1}$$

Let us consider the partial differential equation formally related to (7.1):

$$hu - \lambda^{-1}\mathrm{div}\,(\partial_\xi \phi(x, \nabla u)) = hf, \tag{7.2}$$

$$[\partial_\xi \phi(x, \nabla u) \cdot \nu^\Omega] = 0. \tag{7.3}$$

As for (6.3), this is a symbolic notation. Again, there is a slight abuse of notation when writing (7.2) since the subdifferential of the ϕ-total variation is multivalued.

Definition 7.1. *Let $f \in L^\infty(\Omega)$. We say that $u \in L^2(\Omega, h\,dx)$ is a solution of (7.2)–(7.3) if $u \in BV_\phi(\Omega) \cap L^\infty(\Omega)$ and there is a vector field $\sigma \in \mathcal{A}_\infty(\Omega)$ such that*

(i) *$hu - \lambda^{-1}\mathrm{div}\,(\sigma) = hf$ in $\mathcal{D}'(\Omega)$,*
(ii) *$\phi^0(x, \sigma(x)) \leq 1$ a.e.,*
(iii) *$\int_\Omega \sigma \cdot Du = \int_\Omega |Du|_\phi$,*
(iv) *$[\sigma \cdot \nu^\Omega] = 0$ $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$.*

Thanks to Theorem 4.6 we may also multiply the equation by functions in $BV_\phi(\Omega)$, extending the characterization of the subdifferential given in Theorem 5.1 to the case of the Neumann problem, and proceeding as in Theorem 6.2 we obtain the following theorem.

Theorem 7.2. (i) *Let $f \in L^2(\Omega, hdx)$. There is a unique solution of the problem*

$$\min_{u \in BV_\phi(\Omega) \cap L^2(\Omega, h\,dx)} \mathcal{E}_\lambda(u). \tag{7.4}$$

(ii) *Assume that $f \in L^\infty(\Omega)$. Then there is a unique solution $u \in L^2(\Omega, h\,dx)$ of (7.2)–(7.3). Moreover, the solution u belongs to $L^\infty(\Omega)$, and it minimizes (7.4).*

8. Numerical solution of the PDE. In this section we present an adaptation of Chambolle's algorithm [26] that permits us to solve a discrete version of (6.3) for some particular instances of $\phi(x, \xi)$. Our development will be restricted to the two-dimensional (2D) case, but it can be easily extended to higher dimensions. Let us give some notation that we use in what follows, keeping in mind that, for simplicity, we will denote the discrete functions we use like their continuous counterparts.

Let us consider the discrete domain $\hat{\Omega} = \{0, 1, \ldots, N-1\}^2$ (more generally, we could assume that $\hat{\Omega} \subseteq \{0, 1, \ldots, N-1\}^2$). For convenience, let us denote by $\hat{\Omega}^e$ the extended domain $\{-1, 0, \ldots, N\}^2$. We denote by U the Euclidean space $\mathbb{R}^{(N+2)\times(N+2)}$. Let us give the definition of the discrete gradient which is adapted to problem (6.3) (which considers

Dirichlet boundary conditions). In section 8.2 we shall use Neumann boundary conditions with the definition of the gradient and divergence taken as in [26]. Given $u \in U$, its discrete gradient ∇u will be a vector in $V := U \times U$ given by $(\nabla u)_{i,j} = ((\nabla u)^1_{i,j}, (\nabla u)^2_{i,j})$, $(i,j) \in \hat{\Omega}^e$, where

$$(\nabla u)^1_{i,j} = \begin{cases} u_{i+1,j} - u_{i,j} & \text{if } (i+1,j),(i,j) \in \hat{\Omega}, \\ -u_{i,j} & \text{if } (i+1,j) \notin \hat{\Omega},\ (i,j) \in \hat{\Omega}, \\ u_{i+1,j} & \text{if } (i+1,j) \in \hat{\Omega},\ (i,j) \notin \hat{\Omega}, \\ 0 & \text{if } (i+1,j),(i,j) \notin \hat{\Omega}, \end{cases} \tag{8.1}$$

$$(\nabla u)^2_{i,j} = \begin{cases} u_{i,j+1} - u_{i,j} & \text{if } (i,j+1),(i,j) \in \hat{\Omega}, \\ -u_{i,j} & \text{if } (i,j+1) \notin \hat{\Omega},\ (i,j) \in \hat{\Omega}, \\ u_{i,j+1} & \text{if } (i,j+1) \in \hat{\Omega},\ (i,j) \notin \hat{\Omega}, \\ 0 & \text{if } (i,j+1),(i,j) \notin \hat{\Omega}. \end{cases} \tag{8.2}$$

The above case amounts to saying that $u_{i,j} = 0$ when the indexes are in $\hat{\Omega}^e \setminus \hat{\Omega}$. These definitions of gradient embody the Dirichlet boundary conditions. The extension of the anisotropy ϕ to $\hat{\Omega}^e$ will be made precise in the examples below. The scalar product and the norm in U are defined as usual and denoted by $\langle \cdot,\cdot \rangle_U$ and $\|\cdot\|_U$, but in the absence of ambiguities the subindex will be omitted. In V the scalar product is denoted $\langle p, q\rangle_V = \sum_{i,j\in\hat{\Omega}} p^T_{i,j}\, q_{i,j}$ and the norm $\|p\|_V = \langle p,p\rangle_V$. Finally, the divergence is defined so that it verifies $\langle p, \nabla u\rangle_V = -\langle \operatorname{div} p, u\rangle_U$:

$$(\operatorname{div} p)_{i,j} = \begin{cases} p^1_{i,j} - p^1_{i-1,j} & \text{if } (i,j) \in \hat{\Omega}, \\ 0 & \text{if } (i,j) \notin \hat{\Omega} \end{cases} + \begin{cases} p^2_{i,j} - p^2_{i,j-1} & \text{if } (i,j) \in \hat{\Omega}, \\ 0 & \text{if } (i,j) \notin \hat{\Omega}. \end{cases} \tag{8.3}$$

8.1. Example 1: Geodesic active contour type models. Let us consider the following generalization of the problem studied by Chambolle in [26]:

$$\min_{u\in U,\, u=0 \text{ in } \hat{\Omega}^e\setminus\hat{\Omega}} \frac{\|(u-f)h^{1/2}\|^2_U}{2} + \lambda^{-1} J_g(u), \quad \text{where} \quad J_g(u) = \sum_{(i,j)\in\hat{\Omega}^e} g_{i,j} |(\nabla u)_{i,j}|, \tag{8.4}$$

$f, g, h \in U$ and $h_{i,j} > 0$ for all $(i,j) \in \hat{\Omega}^e$. We consider these functions defined originally in $\hat{\Omega}$ and extended to $\hat{\Omega}^e$ by specular symmetry. Observe that the Euler equation of (8.4) is a discretization of (6.3), where $\phi(x, \nabla u) = g(x)|\nabla u|$ (discrete case), and that where we write $\partial_\xi \phi(x, \nabla u) = g\frac{\nabla u}{|\nabla u|}$,

$$hu - \lambda^{-1} \operatorname{div}\left(g \frac{\nabla u}{|\nabla u|}\right) \ni hf. \tag{8.5}$$

As in [26] let us derive the dual formulation for (8.4) by rewriting (8.5) as $h\lambda(f-u) \in \partial J_g(u)$, which is equivalent to $u \in \partial J^*_g(h\lambda(f-u))$, where J^*_g is the Legendre–Fenchel transform of J_g. Writing $w = \lambda(f-u)$, we have

$$0 \in (w - \lambda f)h + h\lambda \partial J^*_g(hw), \tag{8.6}$$

which is the minimizer of the *dual* problem

$$\min_{w\in U} \frac{\|h^{1/2}w - b\|_U^2}{2} + \lambda J_g^*(hw) \quad \text{with} \quad b = h^{1/2}\lambda f. \tag{8.7}$$

Since J_g is homogeneous, J_g^* is the indicator function of a convex set $\mathcal{K}_g$ given by

$$J_g^*(w) = \begin{cases} 0 & \text{if } w \in \mathcal{K}_g \\ +\infty & \text{otherwise} \end{cases} \text{ with } \mathcal{K}_g = \left\{-\mathrm{div}\,\xi : \xi \in V, |\xi_{i,j}| \le g_{i,j}\ \forall (i,j) \in \hat{\Omega}^e\right\}. \tag{8.8}$$

Therefore we may write (8.7) as

$$\min_{hw\in\mathcal{K}_g} \|h^{-1/2}h\,w - b\|_U^2. \tag{8.9}$$

Note that any solution $hw \in \mathcal{K}_g$, must satisfy $h_{i,j}w_{i,j} = -\mathrm{div}\,(g_{i,j}p_{i,j})$ with $|p_{i,j}| \le 1$. Hence we may write (8.9) as

$$\begin{gathered} \min_{p\in V} \|h^{-1/2}\mathrm{div}\,(gp) + b\|_U^2 \\ \text{subject to} \quad |p_{i,j}|^2 - 1 \le 0 \quad \forall (i,j) \in \hat{\Omega}^e \end{gathered} \tag{8.10}$$

and, introducing the Lagrange multipliers $\alpha_{i,j}$ for the constraint, we obtain the functional

$$\mathcal{F}(p,\alpha) = \sum_{(i,j)\in\hat{\Omega}^e} |h_{i,j}^{-1/2}\mathrm{div}\,(gp)_{i,j} + b_{i,j}|^2 + \sum_{(i,j)\in\hat{\Omega}^e} \alpha_{i,j}(|p_{i,j}|^2 - 1), \quad \alpha \in U,\, p \in V.$$

Proceeding as in [26] the solution of (8.10) satisfies

$$-[g\nabla(h^{-1}\mathrm{div}\,(gp) + \lambda f)]_{i,j} + \alpha_{i,j}p_{i,j} = 0 \quad \forall (i,j) \in \hat{\Omega}^e. \tag{8.11}$$

The Karush–Kuhn–Tucker theorem yields the existence of the Lagrange multipliers $\alpha_{i,j}^* \ge 0$ for the constraints in (8.11), which are either $\alpha_{i,j}^* > 0$ if $|p_{i,j}| = 1$ or $\alpha_{i,j}^* = 0$ if $|p_{i,j}| < 1$, but in this case also $[g\nabla(h^{-1}\mathrm{div}\,(gp) + \lambda f)]_{i,j} = 0$. In any case $\alpha_{i,j}^* = |[g\nabla(h^{-1}\mathrm{div}\,(gp) + \lambda f)]_{i,j}|$, and substituting it into (8.11) and using a gradient descent we arrive at the following fixed-point algorithm:

$$p^{n+1} = \frac{p^n + \tau\left\{g\nabla[h^{-1}\mathrm{div}\,(gp^n) + \lambda f]\right\}}{1 + \tau|g\nabla[h^{-1}\mathrm{div}\,(gp^n) + \lambda f]|}, \tag{8.12}$$

where the maximum $\tau > 0$ will depend on the chosen discretization. For the present scheme, with a straightforward computation [26, 1], one can show that the method converges if $\tau < \frac{1}{8}\frac{1}{\max|g|^2}\frac{1}{\max|h^{-1/2}|^2}$. At convergence, the solution is obtained using the formula $u = f + \lambda^{-1}h^{-1}\mathrm{div}\,(gp)$.

Let us summarize the steps of the algorithm.

Chambolle's algorithm with Dirichlet boundary conditions:

1. Initialize $p^0 = 0 \in V$, $q^0 = 0 \in U$, and $t = 0$
2. Iterate until convergence:
   (a) Compute: $p^{t+1} \leftarrow \frac{p^t + \tau g \nabla q^t}{1+\tau|g\nabla q^t|}$
   (b) Compute: $q^{t+1} \leftarrow h^{-1}\text{div}\,(gp^{t+1}) + \lambda f$
3. Recover the solution $u = \lambda^{-1} q^{t+1}$

In step 2(a) p^t is updated for $(i,j) \in \hat{\Omega}^e$, while in step 2(b) q^t may be updated only for $(i,j) \in \hat{\Omega}$ and $q^t_{i,j} = 0\ \forall (i,j) \in \hat{\Omega}^e \setminus \hat{\Omega}$.

Remark 5. Let us check that the solution u obtained from the fixed point of (8.12) verifies a discrete version of the boundary conditions of Definition 6.1. That is, the field σ satisfies (a discrete version of) $[\sigma, \nu^\Omega] \in \text{sign}(-u(x))\phi(x, \nu^\Omega(x))$, where ν^Ω denotes the outer unit normal to the boundary, and $\phi(x, \nu^\Omega(x)) = g(x)$. The outer unit normals at the points of the discrete boundaries take only four possible values, $\nu^{\hat{\Omega}} \in \{\binom{1}{0}, \binom{-1}{0}, \binom{0}{1}, \binom{0}{-1}\}$; we describe here just one direction (i.e., the left boundary of the domain).

Thus, let us assume that $i = -1$ so that we are at the left boundary side. We check that $[gp, \nu^{\hat{\Omega}}](-1,j) \in \text{sign}(-u_{0,j})\phi((0,j),(-1,0)^t) = \text{sign}(-u_{0,j})g_{0,j}$, which is a discrete way of imposing the boundary condition. Notice that the fixed-point solution p of (8.12) satisfies $g_{-1,j}p_{-1,j}\nabla q_{-1,j} = g_{-1,j}|\nabla q_{-1,j}|$, where $q = h^{-1}\text{div}\,(gp) + \lambda f$ (note as well that $\sigma = gp$). At the left side of the boundary we have $\nabla q_{-1,j} = \binom{q_{0,j}}{q_{-1,j+1} - q_{-1,j}}$ (with $q_{-1,j+1} - q_{-1,j} = 0$ since both pixels are in $\hat{\Omega}^e \setminus \hat{\Omega}$). If $u_{0,j} > 0$, then $q_{0,j} > 0$. Therefore $[gp, \nu^{\hat{\Omega}}](-1,j) = -g_{-1,j}\, q_{0,j}/|q_{0,j}| = g_{0,j}\,\text{sign}(-u_{0,j})$. If $u_{0,j} = 0$, then $[gp, \nu^{\hat{\Omega}}](-1,j) \in g_{0,j}\,\text{sign}(-u_{0,j})$, since $|[gp, \nu^{\hat{\Omega}}](-1,j)| \le g_{0,j}$. The computation for the other three sides of the boundary can be done in a similar way.

8.2. Example 2: An anisotropic diffusion-type problem. In this example we consider an anisotropic diffusion problem with Neumann boundary conditions. The discretization of the gradient and divergence are the same as in [26]. Let us consider the anisotropic total variation with $\phi(x,\xi) = |A_x \xi|$, for all $x \in \Omega$, where A_x is a symmetric and positive definite (hence, invertible) matrix. As before, the solution of the minimization problem

$$\min_{u\in U} \frac{\|h^{1/2}(u-f)\|_U^2}{2} + \lambda^{-1} J_\phi(u) \quad \text{with} \quad J_\phi(u) = \sum_{0\le i,j\le N-1} \phi((i,j), \nabla u(i,j))$$

is obtained via its dual formulation

$$\min_{w\in U} \frac{\|h^{1/2}w - h^{1/2}\lambda f\|_U^2}{2} + \lambda J_\phi^*(hw) \quad \text{with} \quad w = (f-u)\lambda. \tag{8.13}$$

Since J_ϕ is homogeneous, J_ϕ^* is the characteristic function of a set $\mathcal{K}_\phi$ which we will characterize next. Following [5] we have $\mathcal{K}_\phi = \{-\text{div}\,\xi^* : \phi^0(x,\xi^*) \le 1\}$, where

$$\phi^0(x,\xi^*) = \begin{cases} 0 & \text{if } \xi^* = 0, \\ +\infty & \text{if } \xi^* \notin Z_x^\perp, \\ \sup\limits_{\xi:\phi(x,\xi)\le 1} \langle \xi, \xi^* \rangle & \text{if } \xi^* \in Z_x^\perp \setminus \{0\}, \end{cases}$$

with $Z_x = \{\xi : \phi(x,\xi) = 0\} = \{\xi : |A_x\xi| = 0\}$. Since A_x is symmetric and invertible, $Z_x = \{0\}$ and $Z_x^\perp = \mathbb{R}^n$. Since the second condition is empty and $\sup_{\xi:|A_x\xi|\le 1}\langle A_x^{-1}A_x\xi, \xi^*\rangle = |A_x^{-1}\xi^*|$, it holds that $\phi^0(x,\xi^*) \le 1$ if and only if $\xi^* = A_x p$ with $|p| \le 1$. We get that

$$\mathcal{K}_\phi = \{-\mathrm{div}\,\xi^*(x) : \xi^*(x) = A_x p(x), |p(x)| \le 1, \forall x \in \Omega\}.$$

This allows us to write problem (8.13) as

$$\min_{p(x):|p(x)|\le 1} \|h^{-1/2}\mathrm{div}\,(A_x p(x)) + h^{1/2}\lambda f\|_U^2 \tag{8.14}$$

and derive the following fixed-point algorithm:

$$p^{n+1} = \frac{p^n + \tau\left\{A_x\nabla[h^{-1}\mathrm{div}\,(A_x p^n) + \lambda f]\right\}}{1 + \tau|A_x\nabla[h^{-1}\mathrm{div}\,(A_x p^n) + \lambda f]|}. \tag{8.15}$$

At convergence, the solution is obtained using the formula $u = f + \lambda^{-1}h^{-1}\mathrm{div}\,(A_x p)$. For applications to image diffusion it is better to use Neumann boundary conditions, which are imposed by adapting the definitions of the gradient and divergence as in [26].

9. Computation of the ϕ-Cheeger set and numerical aspects of the ϕ-perimeter computation.

The numerical scheme described above produces a function u which is a solution of the PDE (6.3). By Proposition 6.3, the level sets of u are global minima of the anisotropic ϕ-perimeter with an inflating force. And the regional maxima of u are local ϕ-Cheeger sets in a suitable domain containing them.

In this section we describe a method for finding these local ϕ-Cheeger sets. We want to define local extrema of a function $P_\phi(\cdot)/|\cdot|$ which is defined on the set of all connected components of upper level sets of an image u. To fix ideas, let us assume that $N = 2$. Then we have to examine the connected components of the upper level sets $\{u > t\}$, $t \in (0,1]$, of the solution u of (6.3). The ϕ-Cheeger set is defined by $\{u = \|u\|_\infty\}$, but due to the floating-point operations we cannot proceed to a direct computation of this set. Instead we take the ϕ-Cheeger set as the minimum of $t \to P_\phi(\{u > t\})/|\{u > t\}|$ with a suitable discretization of the variable t. Similarly, to compute the local ϕ-Cheeger sets we use the tree of connected components of upper level sets of the image (see [50, 25] and [46] when $N = 3$) and look for the local minima of $P_\phi(cc\{u > t\})/|cc\{u > t\}|$, where $cc\{u > t\}$ denotes a connected component of $\{u > t\}$. Thanks to the topological structure of the tree of connected components of upper level sets, we can speak of local extrema of functions defined on that set. Intuitively, a neighborhood of $\Gamma_t = cc\{u > t\}$ consists in those connected components of upper level sets whose levels are slightly above or below the level of Γ_t.

Let us explain how to compute the weighted perimeter and volume of a given level set. Then we will show how to use this computation to obtain an efficient algorithm to find the connected components of the upper level sets which are local minimizers of the ϕ-Cheeger ratio. When ϕ is of the form $\phi(x,\xi) = g(x)|\xi|$, we will use the expression g-Cheeger set.

9.1. Subpixel computation of weighted perimeters and areas.

Notice that it is not trivial to compute the perimeter of a set which is defined by pixels or voxels. The naive approach of counting the voxels which touch the boundary of the region does not work, mainly because this quantity is not invariant by rotations. There are two common solutions to this problem: approximate the perimeter using integral geometric measure techniques as in graph cuts [12] or approximate the ragged boundary of the set by a smoother surface and compute its perimeter. We found the second option best suited to our needs because, as the goal is to compute perimeters of level sets, we can produce high-resolution approximations of their boundaries by methods such as marching cubes or marching squares [45]. Once we have a triangulated surface, we can compute its weighted perimeter by adding the areas of all triangles, each one multiplied by the weight ϕ interpolated at the barycenter of each triangle.

To test the consistency and precision of this scheme, let us consider a spherical image $u(x) = f(|x - x_0|)$ whose profile f is an increasing function from $[0, +\infty)$ to $[0, 1)$. For each t in $(0, 1)$, the level set of value t is a sphere of radius $r(t) = f^{-1}(t)$ centered at x_0, and this surface is weighted by $\frac{1}{|\nabla u|} = \frac{1}{f'(r(t))} = r'(t)$. The $\frac{1}{|\nabla u|}$-Cheeger ratio is then

$$F(t) = \frac{N V_N r^{N-1} r'}{V_N r^N} = N \frac{r'(t)}{r(t)},$$

where V_N is the volume of the unit ball of $\mathbb{R}^N$. This function $F(t)$ is a real-valued function whose minimum can be evaluated numerically, or even analytically in some easy cases.

We can set, for example,

$$f(x) = \frac{1}{1 + \exp \frac{\kappa - x}{s}}, \tag{9.1}$$

where κ and s are parameters, such as $\kappa = 8$ and $s = 0.1$. Intuitively, the desired segmentation of this image is a circle of radius $\kappa \pm s$ or, equivalently, some level set of value near $t = \frac{1}{2}$. In Figure 1, we compare the graphs of the $\frac{1}{|\nabla u|}$-Cheeger ratio over t as computed analytically and with the numerical methods described above. The minima in both cases is attained very near $t = \frac{1}{2}$, which agrees with our intuition and suggests that the numerical approximations we use are consistent.

As another numerical test, we computed the Euclidean Cheeger set of a square and a cube (using the Euclidean metric). See Figures 2 and 3 for the plausible result we obtained.

The discrete images u are obtained by an iterative numerical method, and they have floating-point values. Most of the values are concentrated around 1, with the interesting part of the range being often contained in the interval $[0.99, 1]$. Thus, it is important to conserve their floating-point values. This implies that there are as many different level surfaces as pixels, one for each different floating-point value. But it is not necessary to compute the ϕ-Cheeger ratio of all of these surfaces: via dichotomic search we can efficiently locate the minimum.

Remark 6. *On the symmetry of the numerical scheme.* In Figure 2 we show an example of the Cheeger set determined with the method described above. Observe that the first solutions are asymmetric; this effect is particularly clear (and annoying) for small images (the first square is 50 pixels high), and it is due to the forward/backward scheme adopted to discretize the gradient (8.1). In [26] the author remarks that the finite difference scheme converges to the continuous formulation as the number of samples $N \to \infty$, but when applied to volumetric images increasing the sampling is not an affordable option. To maintain the symmetry of

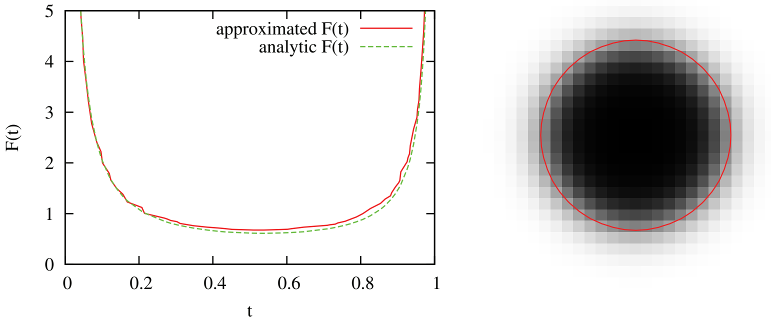
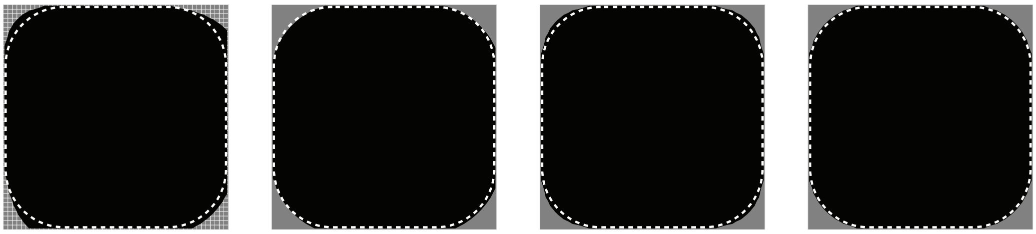


Figure 1. *Numerical evaluation of* $\frac{1}{|\nabla I|}$*-Cheeger ratios for a synthetic image where they can also be computed analytically. The image is given by the model in* (9.1) *with* $\kappa = 8$ *and* $s = 1$*. Left: graphs of the exact and computed* $F(t)$ *for that image. Right: interpolated level curve at which the minimum of* $F(t)$ *is attained, overlaid on the original image. This example has a very low resolution (the original image has dimensions* 31×31*). For higher resolutions the approximation is nearly perfect, and the two curves* $F(t)$ *are visually identical.*



12.858 12.960 12.949 12.961

Figure 2. *Computation of the Euclidean Cheeger sets (black regions) for a square of* 50 *pixels. For the square the analytical Cheeger set is known (dashed line), and its constant is* 1/13.253 *(for readability we display the inverse of the Cheeger constant). The first column shows the results obtained with Chambolle's numerical scheme* [26]*; observe that it is asymmetric. The second solution is obtained by increasing the sampling (as proposed by Chambolle); this considerably improves the solution, but the asymmetry persists. The third column shows a symmetric solution obtained by averaging all the numerical schemes (4 in two dimensions). The fourth solution is obtained using the* consensus algorithm*. All the results were obtained after* 20000 *iterations of* (8.12)*.*

the solutions, we propose the *consensus algorithm*, which computes the mean solution of all the finite difference schemes (4 schemes in two dimensions, and 8 in three dimensions) at each iteration. The consensus algorithm outperforms the considered schemes while keeping the symmetry, but it is 4 times (or 8 for three dimensions) slower than the standard finite difference scheme.

10. Computing the minimum of the geodesic active contour model with an inflating force. Given an image $I : \Omega \to \mathbb{R}$, let us consider the following formulation of geodesic active contour with an inflating force:

$$\min_{E \subseteq \Omega} P_g(E) - \mu |E|_h, \qquad \mu > 0, \tag{10.1}$$

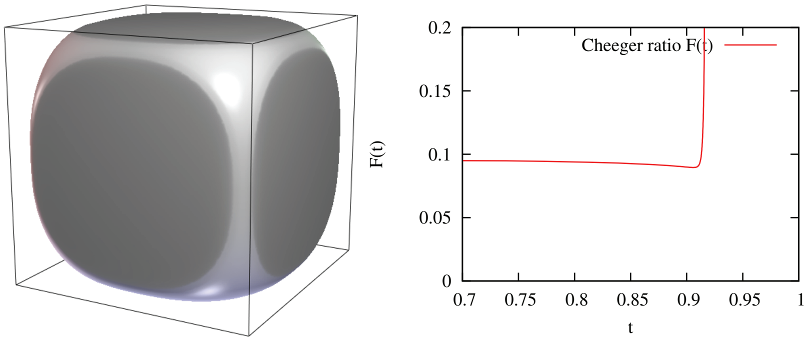


Figure 3. *Cheeger set of a cube. This was computed by Euclidean total variation minimization of the characteristic function of the cube, followed by the selection of the level surface that minimized the Cheeger ratio. On the right, the graph of $F(t)$. Notice that for large values of t, the level set collapses to a central point of the domain, so that its Cheeger ratio is much larger than the optimum.*

where $P_g(E)$ is a weighted perimeter with weights $g(x) = (\sqrt{1+|\nabla(G*I)|^2})^{-1}$, $|E|_h = \int_E h(x)\,dx$ is the weighted area, and μ is a parameter that controls the balloon force. In Proposition 6.3 we have shown that if u is the solution of problem (6.4), then $E_s := \{u \geq s\}, s \in (0,1]$, is a global minimum of (10.1) with $\mu = \lambda(1-s)$. In particular, if $\lambda > \mathcal{C}_\Omega$ (the g-Cheeger constant), then the set $\{u = \|u\|_\infty\}$ is the g-Cheeger set of Ω. Therefore to compute the solutions of (10.1) it suffices to solve problem (6.4) for some $\lambda > \mu$ or, equivalently, to find u by solving

$$hu - \lambda^{-1}\mathrm{div}\left(g\frac{Du}{|Du|}\right) = h. \tag{10.2}$$

The solution of (10.2) is computed using the scheme proposed by Chambolle [26] and described in section 9 (with $f = \chi_\Omega$). For λ big enough, for all values of $\mu > 0$, the solutions of (10.1) can be found as the level sets of u. In practice we select the g-Cheeger set as the upper level set of u that minimizes

$$\min_{\Gamma \subseteq \Omega} \frac{P_g(\Gamma)}{|\Gamma|_h},$$

where the minimum is taken over the upper level sets of u.

10.1. Experiments (segmentation and edge linking). We have used the theory above in two different ways, corresponding to different choices of a metric integrand g. The first choice is $g(x) = (\sqrt{1+|\nabla(G*I)|^2})^{-1}$, and the second choice is the distance function to the set of edge points detected by a preprocessing of the image, that is, $g = d_S$, where S is the set of edges of u. We label these two cases $\frac{1}{|\nabla I|}$-*Cheeger sets* and d_S-*Cheeger sets*, respectively. We observe that the convergence of the iterative scheme to solve the PDE is much faster for d_S-Cheeger sets, and the result is less likely to miss parts of the image. On the other hand, the computation of $\frac{1}{|\nabla I|}$-Cheeger sets gives smooth results after a long time and sometimes misses parts of the desired objects or fails to break at holes. The choice of a subdomain $B \subseteq \Omega$ allows for some flexibility: we can enforce hard restrictions on the result by removing from the domain some points that we do not want to be enclosed by the output surfaces.

Notice that, for a given choice of g, we actually find many *local* g-Cheeger sets, disjoint from the global minimum, that appear as local minima of the g-Cheeger ratio on the tree of connected components of upper level sets. The computation of those sets is partially justified by Proposition 6.11. Notice that the assumptions in it do not cover the case where g vanishes. These are the sets which we show in the following experiments.

2D images. In Figure 4, we display some local g-Cheeger sets of 2D images for different choices of metric g. These experiments are equivalent to applying the model (10.1) to edge linking problems. As in [30] the inflating force allows us to link the pieces of the boundaries of the objects. We display in Figure 4 some 2D linking experiments, which show how the d_S-Cheeger set indeed links the edges. Let us point out here a limitation of this approach, which can be observed in the last subfigure. Even if this linking is produced, the presence of a bottleneck (bottom right subfigure) causes the d_S-Cheeger set to be a set with large volume. This limitation can be circumvented by adding barriers in the domain Ω.

Synthetic 3D image. The first 3D example is a synthetic image built in the following way. We have taken the characteristic function of a slanted torus plus a linear function and then added some blurring and Gaussian noise to the result. Some slices and a level surface of this image are shown in the left subfigure of Figure 5. The first experiment with this synthetic image has been to segment it using the $\frac{1}{|\nabla I|}$-Cheeger set of the image domain. This gives a reasonable segmentation of the object, as shown in Figure 5. The second experiment with this synthetic image has been to perform edge linking. We have taken the output of an edge detector [31, 47] and used the distance function to the set of edges as a metric. The d_S-Cheeger set of the image domain is a surface that correctly interpolates the given patches. We can observe that the result of the edge linking has a ragged appearance. In Figure 6 we display the input edges, the corresponding metric, and the final result. In Figure 7 we display the graph of the $\frac{1}{|\nabla I|}$-Cheeger ratio and different level sets of u.

Real 3D computed tomography (CT) image. The first real 3D example is based on a CT of cerebral arteries containing an aneurysm. We have tried both $\frac{1}{|\nabla I|}$ and d_S metrics (where S is computed, as before, by an edge detector). The results are visually similar. Noticing that both methods give an incorrect segmentation on a small part of the image (at the neck of the aneurysm), we have forced a correct segmentation by manually marking some voxels, as in the rightmost column in Figure 4. Thus, instead of computing the ϕ-Cheeger set of the image domain, we have computed the ϕ-Cheeger set of the image domain minus some manually selected voxels. In Figure 8 we display the results, and in Figure 9, we display three different level surfaces of the solution u (the central one being the $\frac{1}{|\nabla I|}$-Cheeger set).

Real 3D magnetic resonance (MR) image. The second real 3D example is an edge linking experiment coming from an MR image. This is a very low-resolution image, where the thin vessels have a width of one voxel. An edge detector correctly finds most of the vessels (in several different connected components). We show the best six local d_S-Cheeger sets of this image in Figure 10.

11. Anisotropic diffusion applications. Consider the anisotropic diffusion problem formulated as

$$\min_{u\in X} \frac{\|\pi_Z u - f\|^2}{2} + \lambda^{-1} J_\phi(u), \tag{11.1}$$

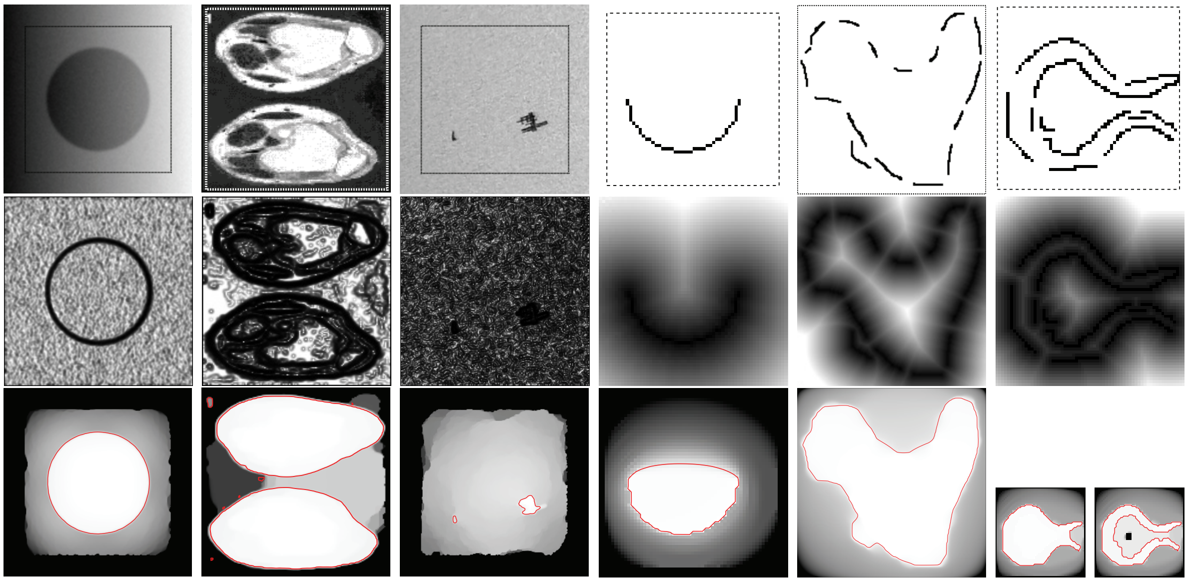


Figure 4. *Geodesic active contours as g-Cheeger minimizers. The first row shows the images I to be processed. The second row shows the weights g used for each experiment (white is 1, black is 0): in the first two cases $g = (\sqrt{1 + |\nabla(G * I)|^2})^{-1}$, for the third $g = 0.37(\sqrt{0.1 + |\nabla(G * I)|^2})^{-1}$, and for the linking experiments $g = d_S$, the scaled distance function to the given edges. The third row shows the disjoint minimum g-Cheeger sets extracted from u (shown in the background); there are $1, 7, 2, 1, 1,$ and 1 sets, respectively. The last linking experiment illustrates the effect of introducing a barrier in the initial domain (black square).*

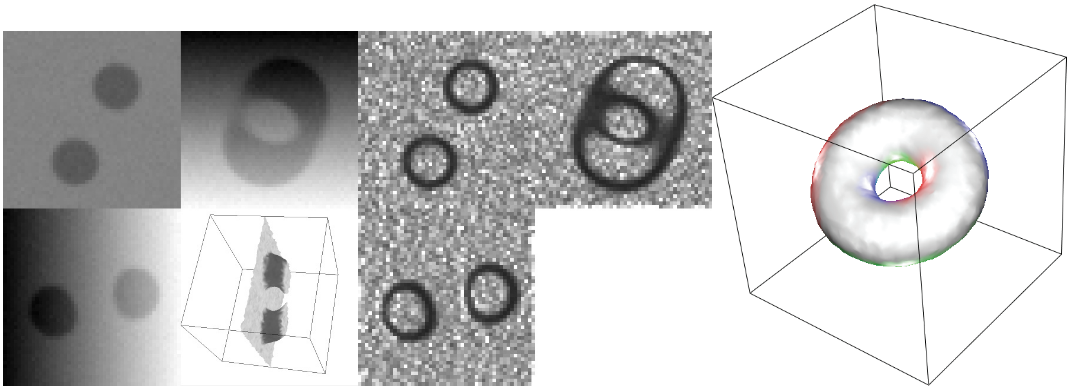


Figure 5. *Pipeline for computing $\frac{1}{|\nabla I|}$-Cheeger sets, applied to a synthetic 3D image. From left to right: slices of the original image I, slices of the metric $g = \frac{1}{\|\nabla I\|}$, and $\frac{1}{|\nabla I|}$-Cheeger set of the image domain.*

where π_Z is the orthogonal projection onto a set $Z \subset X$, and $f \in Z$. The regularizer $J_\phi(u) = \int_\Omega \phi(x, \nabla u(x))$ is defined so that the diffusion is constrained to the geometry (given by the level lines) extracted from a reference image I. We define, for example, $\phi(x, \xi) = |A_x \xi|$, where A_x is a matrix that embodies knowledge about the boundaries of the objects in I. A common example in two dimensions corresponds to $A_x = V(x)^\perp \otimes V(x)^\perp$ with $V(x) = \frac{\nabla I(x)}{\sqrt{1+|\nabla I(x)|^2}}$. This example favors the diffusion along the level lines of I. In low gradient (flat) zones

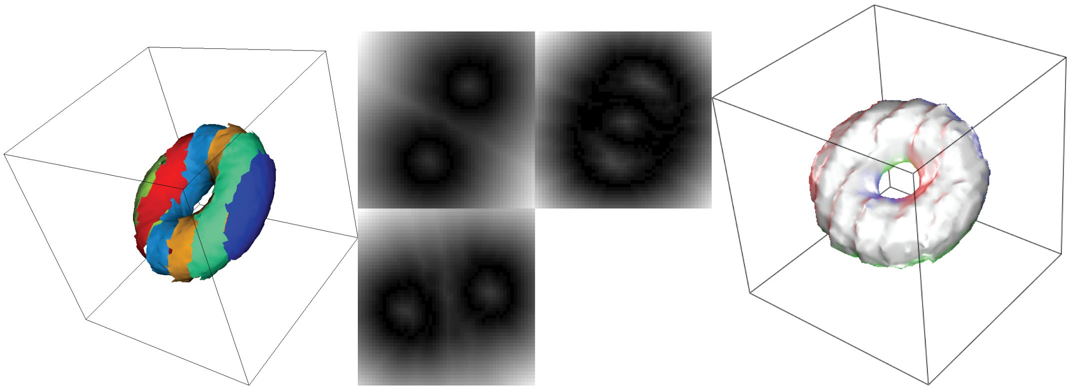


Figure 6. *Pipeline for computing d_S-Cheeger sets, applied to the same synthetic image as in Figure 5. From left to right: detected 3D edges S, slices of the metric $g = d_S$, and d_S-Cheeger set of the image domain.*

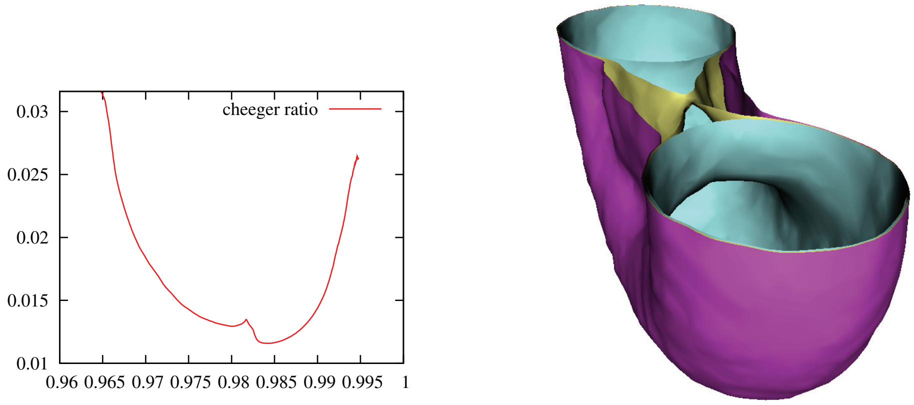


Figure 7. *Left: graph of the $\frac{1}{|\nabla I|}$-Cheeger ratio $F(t)$ for the input image "torus." Right: superposition of the three level sets shown, corresponding to the interesting points of $F(t)$ (the two minima and the cusp). To see the inner surfaces, the display is clipped near the central singularity. Notice that these level surfaces are all local minima of the classical geodesic snakes functional with an inflating force, for different weights of the inflating force.*

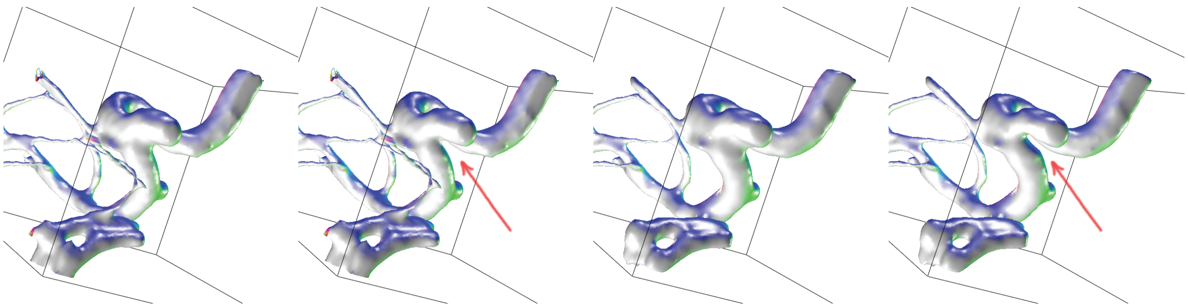


Figure 8. *Computation of $\frac{1}{|\nabla I|}$-Cheeger sets of the CT image. From left to right: (1) d_S-Cheeger set of the whole image domain, (2) d_S-Cheeger set of the image domain minus some manually selected voxels at the neck of the aneurysm, (3) $\frac{1}{|\nabla I|}$-Cheeger set of the whole image domain, and (4) $\frac{1}{|\nabla I|}$-Cheeger set of the image domain minus some manually selected voxels at the neck of the aneurysm.*

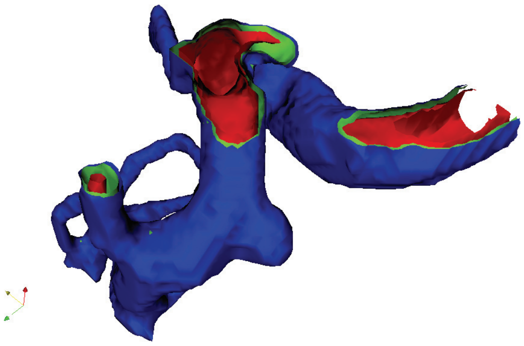


Figure 9. *Three level surfaces of the solution u of $\frac{1}{|\nabla I|}$-total variation minimization. The central surface is the $\frac{1}{|\nabla I|}$-Cheeger set of the image domain according to this active–contours-like metric. The other two surfaces have a higher $\frac{1}{|\nabla I|}$-Cheeger ratio and appear as local extrema of an active contour with appropriate inflating force. In this figure, the image domain is split so that the innermost surfaces can be seen. Notice that the inner surface, having a higher level t, is separated from the other two. This indicates the concentration of values around the maximum 1.*

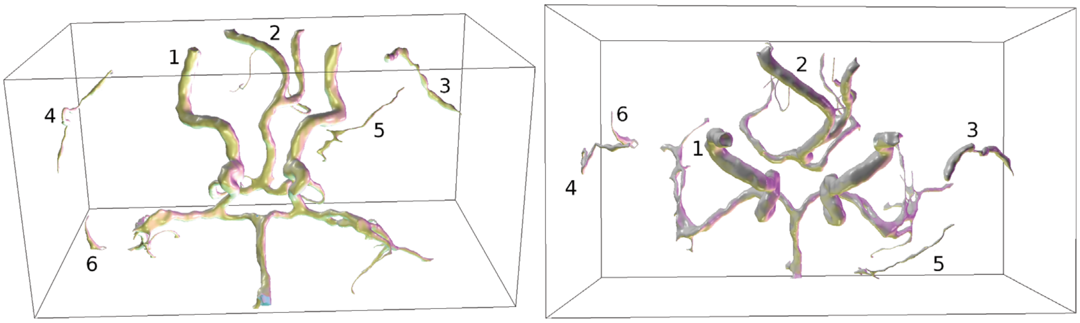


Figure 10. *These two figures display the best six local d_S-Cheeger sets of the MR image, labelled and from different points of view.*

the previous definition can be relaxed to allow diffusion across the level lines (as depicted in Figure 11) in a way inversely proportional to the modulus of the gradient. In that case, we may take $A_x = V(x)^\perp \otimes V(x)^\perp + \frac{1}{\sqrt{1+|\nabla I(x)|^2}} V(x) \otimes V(x)$, where $V(x)^\perp$ denotes the counterclockwise rotation of $V(x)$ of angle $\frac{\pi}{2}$. Notice that by the structure of A_x we could also take the clockwise rotation.

We will solve (11.1) by adapting the zoom algorithm proposed in [26]. Observing that $\|\pi_Z u - f\| = \min_{w \in Z^\perp} \|u - (f + w)\|$, (11.1) can then be reformulated as

$$\min_{u \in X, w \in Z^\perp} \frac{\|u - f - w\|^2}{2} + \lambda^{-1} J_\phi(u), \tag{11.2}$$

which is solved by alternate minimization with respect to u and w. The first minimization is done by the algorithm described in section 8.2, $u_n = (f - w_n) - \pi_{\mathcal{K}_\phi}(f - w_n)$, and the second one consists in a projection over $Z^\perp$: $w_{n+1} = \pi_{Z^\perp}(u_n - f)$.

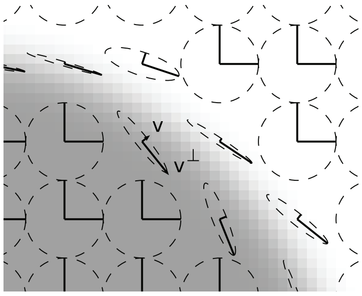


Figure 11. *Anisotropic diffusion directions for a synthetic image (background). The plotted ellipses correspond to the tensor field* $A_x = V(x)^\perp \otimes V(x)^\perp + \frac{1}{\sqrt{1+|\nabla I(x)|^2}} V(x) \otimes V(x)$ *with* $V(x) = \frac{\nabla I(x)}{\sqrt{1+|\nabla I(x)|^2}}$*. The orientation and widths of the ellipses reflect the preferred direction of the diffusion.*

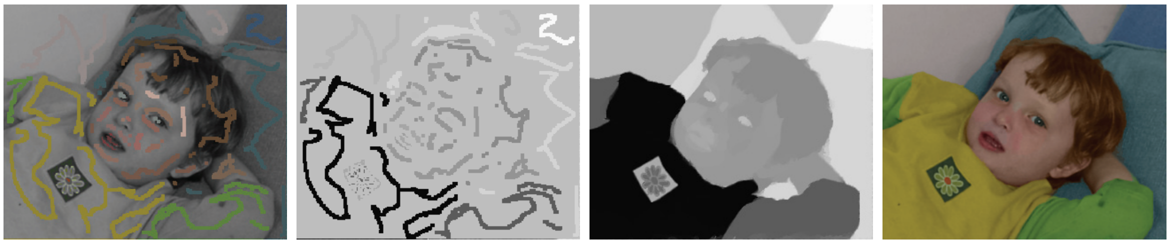


Figure 12. *Colorization. From left to right, scribbled image, the known values of the U channel (20% of the image), diffusion in the U channel, and colorized result using the reference Y channel (not shown).*

11.1. Experiments (diffusion). The scheme presented earlier for solving (11.2) can be applied in a variety of diffusion problems, such as image colorization [44], or to the interpolation of sparse height data in a digital elevation model [33]. In each of these cases, however, there are better algorithms for performing the task than the one we propose here, which is meant only as an illustration.

In the case of colorization and interpolation, Z is defined as $Z = \{\chi_\Gamma f : f \in X\}$, where $\Gamma \subseteq \Omega$ is a subdomain of the image where the values are known, and the reference image $I : \Omega \to \mathbb{R}$ is used to compute the field $V(x)$ to guide the diffusion of these values. For the colorization experiment shown in Figure 12 the result is computed in YUV color space, where Y is the input luminance channel and the chromatic channels U and V are interpolated with (11.1), where the field $V(x) = \frac{\nabla I(x)}{\sqrt{1+|\nabla I(x)|^2}}$ restricts the diffusion to the geometry of I.

The last example concerns the interpolation of urban digital elevation models (see Figure 13). In this case the datum f is known only at sparse locations, and it is provided by a stereo subpixel correlation algorithm [52] (which also provides an estimation of the measure's variance Err). The reference image of the stereo pair is used as a geometric constraint for the interpolation, and the variability Err is used to normalize the data fitting by adapting the spatial metric $h(i,j)^{1/2} = 1/Err(i,j)$. In Figure 13 we compare this method with the anisotropic minimal surface interpolation described in [33].

12. Conclusion. We have developed the mathematical analysis of ϕ-total variation problems with eventually degenerate metric integrands ϕ. As a particular case, we have considered

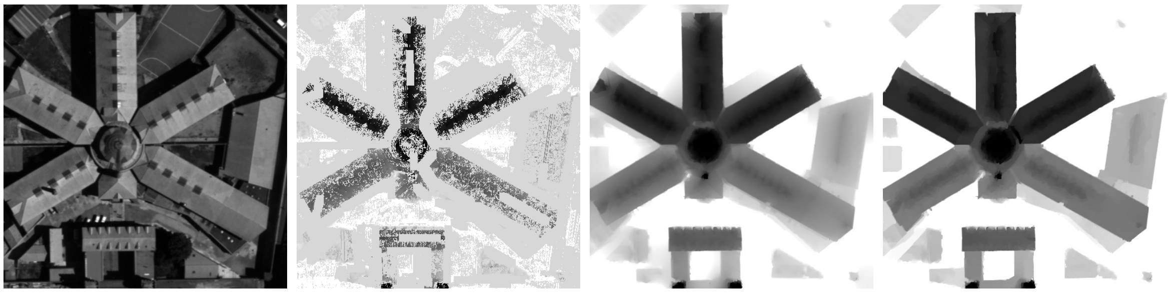


Figure 13. *Disparity interpolation in an urban digital elevation model. From left to right:* (1) *the reference image of the stereo pair,* (2) *the incomplete data set computed with* [52] *(*30% *of the image), where each point's gray level represents the height (darker is higher, mid-gray is unknown),* (3) *the interpolation obtained with the minimal surface interpolation* [33] *(RMSE* 0.239 *when compared with the ground truth) and with the proposed algorithm (RMSE* 0.190*),* (4) *the minimal surface model* [33] *recovers the slanted surfaces better than total variation; however, the latter is better at approximating the geometry near jumps.*

the geodesic active contour model, which corresponds to $\phi(x,\xi) = g(x)|\xi|$, where $g(x)$ is a function that may vanish for some values of x. We have defined the notion of ϕ-Cheeger set and we have shown that, for suitable metric integrands ϕ, the maximal ϕ-Cheeger set can be computed as the level set associated to the maximum of the solution of a ϕ-total variation minimization problem with Dirichlet boundary conditions and datum $f = 1$. We have also defined the notion of local ϕ-Cheeger set. Moreover, the level sets of the solution of the ϕ-total variation minimization problem with Dirichlet boundary conditions are global minimizers of the ϕ-perimeter with an inflating force. Thus, in the particular case of the geodesic active contour model with inflating force, we can compute a global minimum. Moreover, the model can be used for edge linking or to interpolate data along the level lines of a reference image.

Appendix A.

A.1. The Borel measure associated to the ϕ-total variation. Let $\mathcal{G}(\Omega)$ denote the family of open sets of Ω. To prove that $U \in \mathcal{G}(\Omega) \to |Du|_\phi(U)$ is an inner content we first check the following properties:

(i) If $U \subseteq V$ are open sets, then $|Du|_\phi(U) \le |Du|_\phi(V)$.

(ii) If U, V are open sets such that $U \cap V = \emptyset$, then $|Du|_\phi(U \cup V) = |Du|_\phi(U) + |Du|_\phi(V)$.

(iii) If U, V are open sets, then $|Du|_\phi(U \cup V) \le |Du|_\phi(U) + |Du|_\phi(V)$.

Let us just check (iii). Recall that, according to (3.7),

$$|Du|_\phi(U \cup V) := \sup\left\{ \int_{U\cup V} u \operatorname{div}\sigma \, dx : \sigma \in \mathcal{K}_\phi^{\infty,c}(U \cup V) \right\}.$$

If $\sigma \in \mathcal{K}_\phi^{\infty,c}(U \cup V)$ has support contained in the compact set $K \subseteq U \cup V$, and φ_1, φ_2 is a partition of unity on K subordinated to U, V, then $\sigma = \sigma\varphi_1 + \sigma\varphi_2$, $\sigma\varphi_1 \in \mathcal{K}_\phi^{\infty,c}(U)$, $\sigma\varphi_2 \in \mathcal{K}_\phi^{\infty,c}(V)$. Hence

$$\int_{U\cup V} u \operatorname{div}\sigma \, dx = \int_U u \operatorname{div}(\sigma\varphi_1)\, dx + \int_V u \operatorname{div}(\sigma\varphi_2)\, dx \le |Du|_\phi(U) + |Du|_\phi(V).$$

Taking sup in σ, we obtain (iii).

Now for any compact set $K \subseteq \Omega$, we define

$$\Lambda(K) := \inf\{|Du|_\phi(U) : K \subseteq U\}.$$

Then Λ is a content measure on the family $\mathcal{F}_K$ of compact sets of Ω (see [36]). The content Λ defines an inner content

$$\Lambda^*(U) := \sup\{\Lambda(C) : C \in \mathcal{F}_K, C \subseteq U\}, \qquad U \in \mathcal{G}(\Omega).$$

Notice that $\Lambda^*(U) = |Du|_\phi(U)$ for any $U \in \mathcal{G}(\Omega)$. Indeed we may take an increasing family of compacts sets K_n such that $\cup_n \mathrm{int}(K_n) = U$. Then $|Du|_\phi(\mathrm{int}(K_n)) \le \Lambda(K_n) \le |Du|_\phi(U)$. We deduce that

$$\sup_n |Du|_\phi(\mathrm{int}(K_n)) = \sup_n \Lambda(K_n) = |Du|_\phi(U).$$

Since $\sup_n \Lambda(K_n) = \Lambda^*(U)$, we have $\Lambda^*(U) = |Du|_\phi(U)$.

A.2. Proof of the results of section 4.

Proof of Proposition 4.1. Case $z \in X_\infty(\Omega)$. Take $\varphi \in \mathcal{D}(U)$ and consider an open set V such that $\mathrm{supp}(\varphi) \subset V \subset\subset U$. Observe that $\mathrm{div}(z\varphi) = \varphi \mathrm{div}\, z + z \cdot \nabla\varphi$ in $\mathcal{D}'(\Omega)$ and $\mathrm{div}(z\varphi) \in L^\infty(\Omega)$. Hence, by definition of ϕ-total variation, we have

$$\begin{aligned} |\langle z \cdot Du, \varphi\rangle| &= \left|\int_\Omega u \mathrm{div}\,(z\varphi)\, dx\right| = \|\varphi\phi^0(x,z)\|_\infty \left|\int_\Omega u\,\mathrm{div}\left(\frac{z\varphi}{\|\varphi\phi^0(x,z)\|_\infty}\right) dx\right| \\ &\le \|\varphi\phi^0(x,z)\|_\infty |Du|_\phi(U) \le \|\varphi\|_\infty \|\phi^0(x,z)\|_{L^\infty(U)} |Du|_\phi(U) \end{aligned}$$

since $\frac{z\varphi}{\|\varphi\phi^0(x,z)\|_\infty}$ has compact support in U. ■

As a consequence of Proposition 4.1, the following result holds.

Corollary A.1. *Assume that $z \in \mathcal{A}_\infty(\Omega)$. Let $|z \cdot Du|$ be the measure total variation of $z \cdot Du$. The measures $z \cdot Du$, $|z \cdot Du|$ are absolutely continuous with respect to the measure $|Du|_\phi(\cdot)$ and*

$$\left|\int_B z \cdot Du\right| \le \int_B |z \cdot Du| \le \|\phi^0(x,z)\|_{L^\infty(U)} |Du|_\phi(B)$$

for all Borel sets B and for all open sets U such that $B \subset U \subset \Omega$.

Lemma A.2. *Assume $u \in BV_\phi(\Omega)$, $z \in \mathcal{A}_\infty(\Omega)$. Let $u_n \in W^{1,1}(\Omega)$ be a sequence converging to u in $L^1(\Omega)$ and such that $\int_\Omega \phi(x, \nabla u_n) \to \int_\Omega |Du|_\phi$. Then we have*

$$\int_\Omega z \cdot \nabla u_n\, dx \to \int_\Omega z \cdot Du.$$

Proof. For a given $\epsilon > 0$, we take an open set $U \subset\subset \Omega$ such that

$$|Du|_\phi(\Omega \setminus U) < \epsilon.$$

Moreover, since the family of measures $\phi(x, \nabla u_n)$ is also bounded, we may assume that it is weakly* convergent to a measure μ. Assume that U is chosen so that $\mu(\Omega \setminus U) < \epsilon$. Let $\varphi \in \mathcal{D}(\Omega)$ be such that $\varphi(x) = 1$ in U and $0 \le \varphi \le 1$ in Ω. Then

$$\left|\int_\Omega z \cdot Du_n - \int_\Omega z \cdot Du\right|$$

$$\leq |\langle z \cdot Du_n, \varphi \rangle - \langle z \cdot Du, \varphi \rangle| + \int_\Omega |z \cdot Du_n|(1-\varphi) + \left| \int_\Omega z \cdot Du(1-\varphi) \right|.$$

Since

$$\lim_{n\to\infty} \langle z \cdot Du_n, \varphi \rangle = \langle z \cdot Du, \varphi \rangle,$$

$$\limsup_{n\to\infty} \int_\Omega |z \cdot Du_n|(1-\varphi) \leq \|\phi^0(x, z(x))\|_\infty \limsup_{n\to\infty} \int_{\Omega\setminus U} \phi(x, \nabla u_n)$$
$$\leq \|\phi^0(x, z(x))\|_\infty \mu(\Omega \setminus U) < \epsilon \, \|\phi^0(x, z(x))\|_\infty,$$

$$\left| \int_\Omega z \cdot Du(1-\varphi) \right| \leq \int_\Omega |z \cdot Du|(1-\varphi) \leq \epsilon \, \|\phi^0(x, z)\|_\infty,$$

and ϵ is arbitrary, the lemma follows. ∎

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