tutorial on graph-cut metrics

enric meinhardt llopis

17–5–2007

Outline

Intro: computing lengths in binary images

Cauchy-Crofton formula

Graph-Cuts

Epilogue: premature discretization?

Outline

Intro: computing lengths in binary images

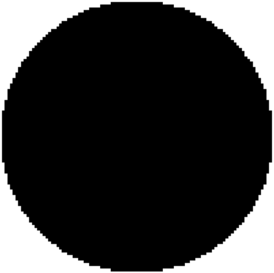
Cauchy-Crofton formula

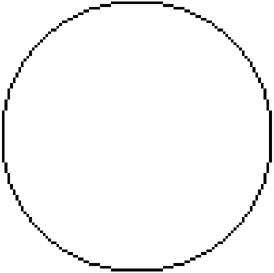
Graph-Cuts

Epilogue: premature discretization?

How to measure the boundary of a region of pixels?

Idea: count the number of pixels on the boundary



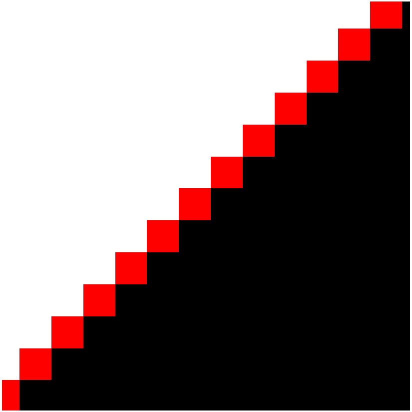


circle of diameter 100

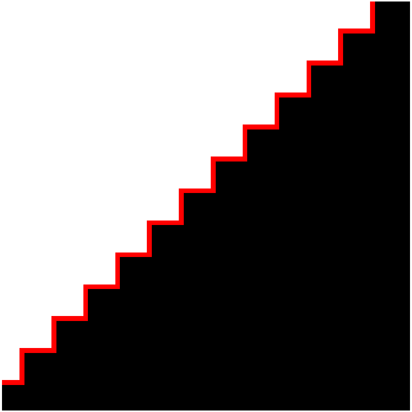
280 boundarying pixels ($\neq$ 314)

How to measure the boundary of a region of pixels?

What shall we count, pixels or edgels?



pixels



edgels

Outline

Intro: computing lengths in binary images

Cauchy-Crofton formula

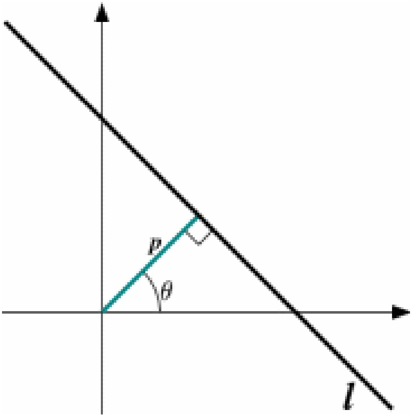
Graph-Cuts

Epilogue: premature discretization?

Crofton formula

$$\mathrm{length}(C) = \frac{1}{2}\int_0^{2\pi}\int_0^{\infty} n_C(\rho,\theta)\,d\rho\,d\theta$$

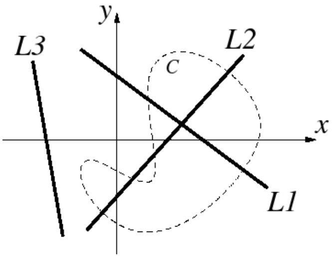
where $n_C(\rho,\theta)$ is the number of intersections of C with the straight line (ρ,θ).



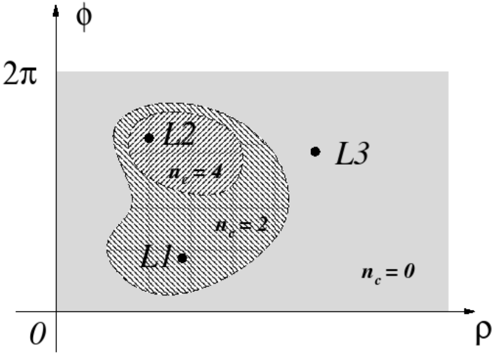
Crofton formula

$$\mathrm{length}(C) = \frac{1}{2}\int_0^{2\pi}\int_0^{\infty} n_C(\rho,\theta)\,d\rho\,d\theta$$

where $n_C(\rho,\theta)$ is the number of intersections of C with the straight line (ρ,θ).



(a) Lines in R^2.



(b) Lines as points in $\mathcal{L}$.

Example of Crofton's formula

$$\text{length}(C) = \frac{1}{2}\int_0^{2\pi}\int_0^{\infty} n_C(\rho,\theta)\,d\rho\,d\theta$$

The perimeter of a circle of radius R is $2\pi R$, because:

- The straight lines that intersect the circle are those whith $\rho \leq R$
- In the (ρ, θ) plane they form a rectangle of area $2\pi R$

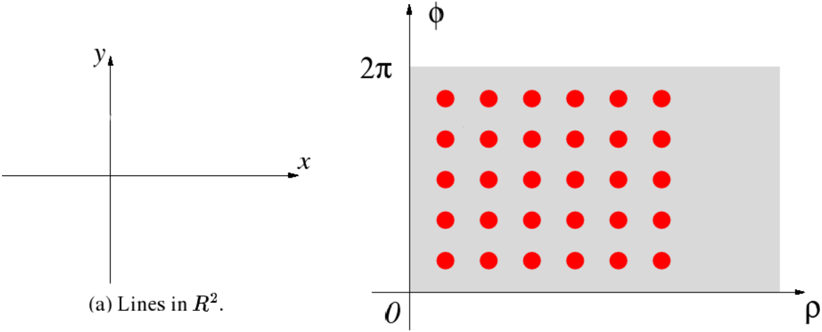
Proof of Crofton's formula

$$\text{length}(C) = \frac{1}{2}\int_0^{2\pi}\int_0^{\infty} n_C(\rho,\theta)\,d\rho\,d\theta$$

- The integral above is additive over unions of curves
- It is invariant under rotations and translations
- The lenght of segments coincides

Discretization of Crofton's formula

Naïve discretization of the (ρ, θ) space: for each line count its number of intersections with our curve

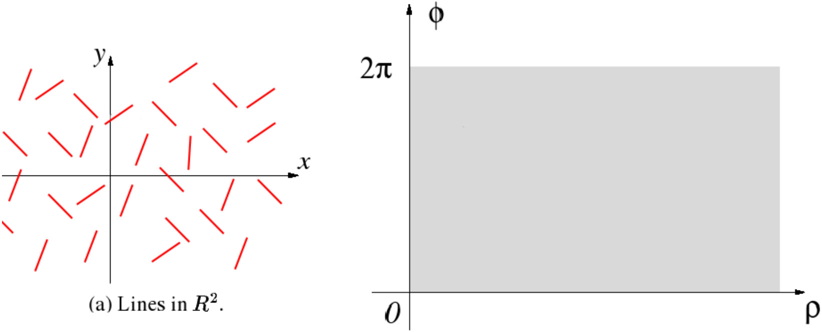


(a) Lines in R^2.

(b) Lines as points in $\mathcal{L}$.

Alternative discretization of Crofton's formula

Smart discretization: count how many "edgels" our curve crosses

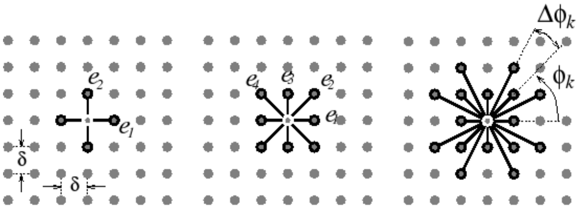


(a) Lines in R^2. (b) Lines as points in $\mathcal{L}$.

Outline

Images as graphs

Idea: represent the image domain as a grid graph with a lot of connectivity



(a) 4 n-system (b) 8 n-system (c) 16 n-system

Images as graphs

Common operations on images represented as graphs:

- Areas: Count the number of nodes inside a region
- Lengths: Count the number of edges cut by a cuve (edges are weighted in accordance with Crofton's formula)
- Closed curve: Cut of the graph

How to assign weights to edges?

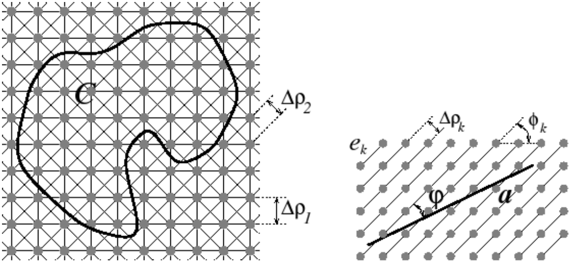
If

$$w_k = \frac{\delta^2 \cdot \Delta\phi_k}{2 \cdot |e_k|}$$

then

$$\mathrm{length}(C) = \sum_k n_c(k) \cdot w_k$$

is a good approximation of euclidean length.



(a) 8-neighborhood 2D grid (b) One family of lines

Non-euclidean metrics

Why non-euclidean metrics?

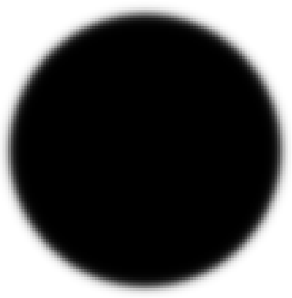
- They are good to rescue people (sand/water refraction)
- They are good to detect edges

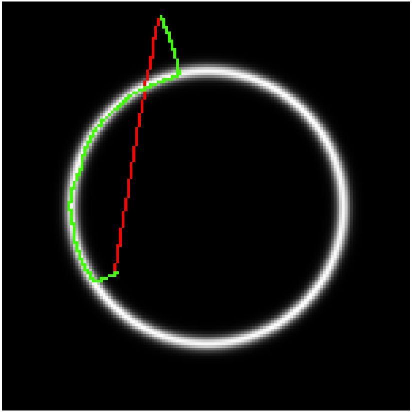
How to compute them using graph-cuts?

- Multiply each edge weight by the metric

Non-euclidean geodesics

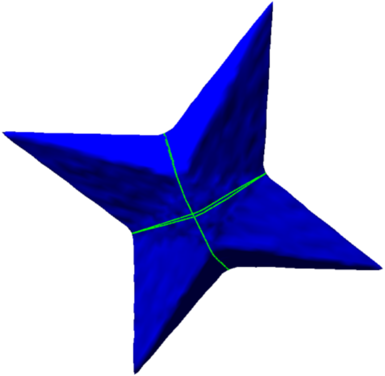
Example of non-uniform metric (a decreasing function of the contrast)





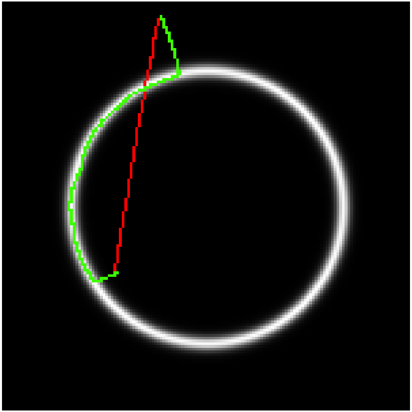
Closed geodesics

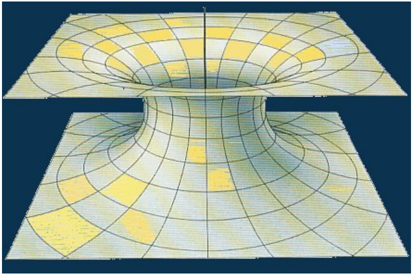
Cuts in graphs represent closed geodesics



Closed geodesics

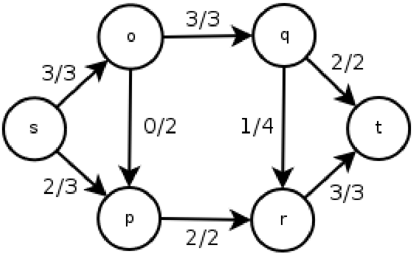
Cuts in graphs represent closed geodesics



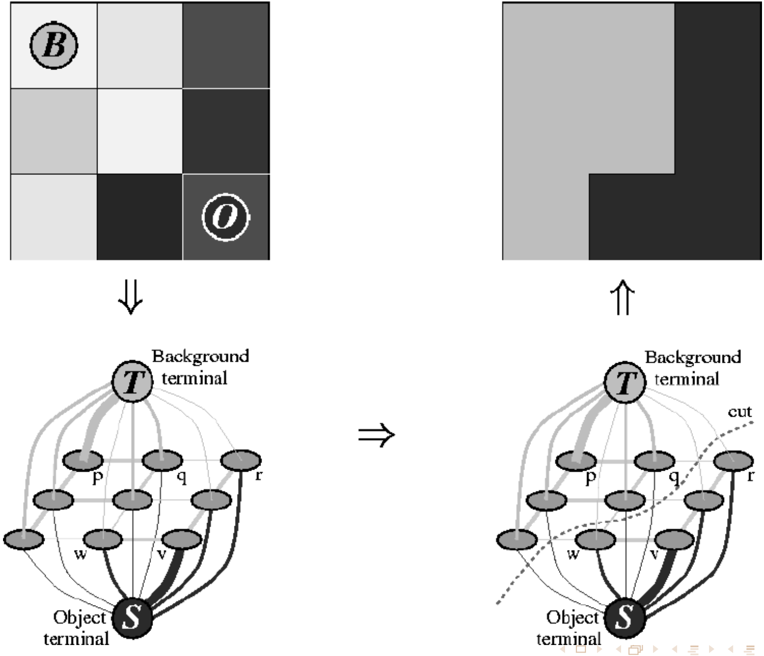


Graph Cuts: How do they work?

- *WW2 optimization problem*: How to inflict the maximum harm in a transportation network using the minimum of bombs?
- *Ford-Fulkerson theorem*: There is a correspondence between minimal cuts and maximum flows.
- To segment an image, build a graph with smart edge weights so that the cost of cuts approximates the functional you want to minimize.



Graph Cuts: How do they work?



Outline

Intro: computing lengths in binary images

Cauchy-Crofton formula

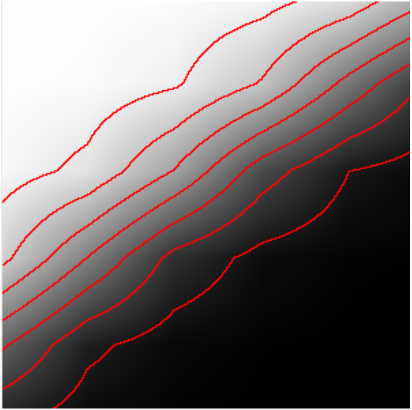
Graph-Cuts

Epilogue: premature discretization?

Computing lenghts of level curves

On interpolated grayscale images, level curves can be sampled at very high precision to give accurate estimations of length.

smooth circle



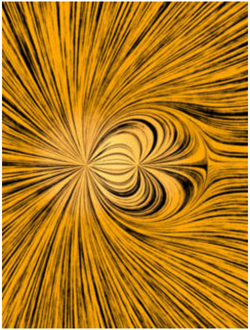
sampled level lines

Parody quote

PREMATURE DISCRETIZATION IS THE ROOT OF ALL EVIL

Continuous model

There is a continuous version of graph cuts. It can be simulated physically or numerically.



- Consider flows as vector fields
- Impose a maximal flow at each point
- Pour a fluid from a source to a sink
- Find the "saturated" curves that separate the source from the sink

References

- For graph cuts: *Boykov & Kolmogorov*
- For Crofton's formula: *Santaló*
- For the continuos version: *Strang*