Hilbert and Fourier analysis
C1

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Today's topics

- Lebesgue integral
  - Sets, numerable union of sets
  - Nonnegative functions. (Beppo-Levi, Fatou)
  - Positive, negative and complex functions
- Function space $L^1(\mathbb{R}^N)$ and convergence theorems
  - Lebesgue's dominated convergence theorem
  - Bounded convergence
- Some results of Integration Theory
  - Density of $\mathcal{C}_c$ in L^1
  - Derivation under the sum
  - Change of variable Theorem
  - Swap integrals (Fubini-Tonelli)

Lebesgue integral

We assume that exists a class of functions $f : \mathbb{R}^N \to [-\inf, \inf]$, which we call **Lebesgue-measurable** functions.

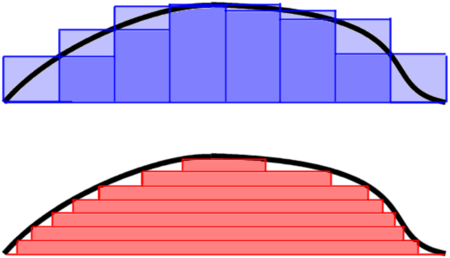
We assume that exists a map, which associates each positive measurable function $f : \mathbb{R}^N \to [0, \inf]$ to its **Lebesgue integral**, we denote it:

$$\int_{\mathbb{R}^N} f(x)dx \quad \text{or} \quad \int f(x)dx \quad \text{or} \quad \int f \quad \in \quad [0, \inf]$$

If $f = \mathbf{1}_E$ (the characteristic function of $E \subset \mathbb{R}^N$) and f is measurable, then we say that the set E is Lebesgue-measurable, and we denote $\boxed{\mu(E) = \int \mathbf{1}_E}$ the **measure of the set** E.

Lebesgue integral

The Lebesgue integral extends Riemann's.



- Ex. allows to compute: $\int_{[0,1]} \mathbf{1}_{\mathbb{Q}} = 0$
- It is less restrictive* for interchanging lim with $\int$

* bounded+pointwise convergence of the sequence, instead of uniform convergence

Lebesgue integral

We assume some **properties** of the Lebesgue integral as axioms

Property 2.1: **(Positive linearity)**
$\forall f, g$ with values in $[0, +\infty]$, and $\lambda, \mu \in \mathbb{R}^+$

$$\int \lambda f + \mu g = \lambda \int f + \mu \int g.$$

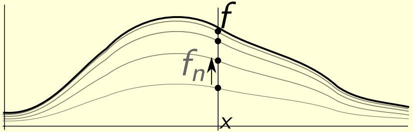
Remark: $0 \leq f \leq g \quad \Rightarrow \quad \int f \leq \int g$. (Hint: use $g = f + \varepsilon$ & $\varepsilon \geq 0$)

Property 2.2: If $f \geq 0$ is **Riemann-integrable**, then it is Lebesgue-measurable and the Lebesgue integral *coincides* with Riemann's.

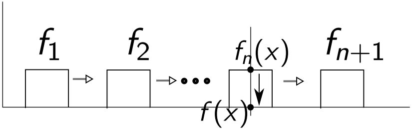
Lebesgue integral

Property 2.3: **(Beppo-Levi or Monotone Convergence Thm)**
If f_n is a monotone increasing sequence of nonnegative ($f_n \in [0, +\infty]$) measurable functions defined on $\mathbb{R}^N$, then

$$\int \underbrace{f(x)}_{\text{pointwise limit}} dx = \int \lim_{n\to\infty} f_n(x) dx = \lim_{n\to\infty} \int f_n(x) dx \leq +\infty$$



Example non monotone: f_n measurable, $f_n(x) \xrightarrow{n\to\infty} 0$ but $\int f > 0$



Lebesgue integral

Exercise 1.

Let $\sum u_n$ be a series of functions $\mathbb{R}^N \to [0, \infty]$.
Show that we can swap the sum and the integral:
$\int \sum_n u_n(x) dx = \sum_n \int u_n(x) dx$.

Proof sketch:

- Define a $S_n(x)$ as a partial sum
- S_n is increasing
- Apply monotone convergence to the sequence
- Apply linearity to the partial sum

Lebesgue integral

Proposition 2.1: Let A, B be measurable sets, then

1. If $A \subset B$, then $\mu(A) \leq \mu(B)$
2. If A_n disjoint, then $\mu(\cup_n A) = \sum_n \mu(A_n)$
3. If A_n not disjoint, then $\mu(\cup_n A) \leq \sum_n \mu(A_n)$

Proof $\rightarrow$ see notes

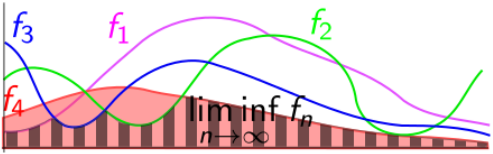
Lebesgue integral

Def : **Limit inferior** of a bounded below sequence $x_n \in \mathbb{R}$ is:

$$\liminf_{j\to\infty} x_j = \lim_{j\to\infty}\left(\inf_{k\geq j} x_k\right) \in (-\infty,\infty]$$

Lemma 2.1: **(Fatou's Lemma)**
Let f_n be a sequence of nonnegative functions. Then

$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n \qquad \text{valid also with } \infty.$$



Proof → blackboard

- Exercise 12: find f_n s.t. $\int \liminf_{n\to\infty} f_n < \liminf_{n\to\infty} \int f_n$ (strict inequality).

Almost Everywhere (a.e.) / *Presque Partout (p.p.)*

Def 2.1: **A set E is negligible** if $\mu(E) = 0$.
A property is said to be valid **almost everywhere (a.e.)** if the set E where it is invalid has $\mu(E) = 0$.

Remark: Numerable union of negligible sets is negligible. (Prop 2.1)

Proposition 2.2: Let $f : \mathbb{R}^N \to [0, +\infty]$, then

$$\int f(x)dx = 0 \Leftrightarrow f(x) = 0 \text{ a.e.}$$

Proof → *blackboard*

Integral of positive, negative and complex functions

Def 2.2:
$f : \mathbb{R}^{\mathbb{N}} \to \mathbb{C}$ is **Lebesgue-integrable** if $\int_{\mathbb{R}^N} |f(x)| dx < \infty$.
And with $\mathcal{L}^1(\mathbb{R}^N)$ we denote the space of integrable functions.

- If $f(x) \in \mathbb{R}$ we can write $f = f^+ - f^-$ where $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$. Then we define:

$$\int f = \int f^+ - \int f^-.$$

- If $f \in \mathbb{C}$ with $f = f_1 + i\, f_2$, then we define:

$$\int f = \int f_1 + i \int f_2.$$

Integral of positive, negative and complex functions

Exercise 2.
Let f be a positive-valued function s.t. $f = g - h$ with $g \geq 0$, $h \geq 0$ with finite integrals. Then $\int f = \int g - \int h$.

Proof hint.
Note that $f = g^+ - h^+$ and also $f = f^+ - f^-$.

Integral of positive, negative and complex functions

Proposition 2.3:

The map $f \mapsto \int f$ **is linear** from $\mathcal{L}^1(\mathbb{R}^N)$ to $\mathbb{C}$.

Let f, g be complex valued and **integrable functions**, then

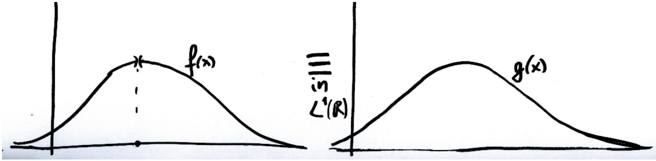
$$\left| \int f(x) dx \right| \leq \int |f(x)| \, dx, \quad \text{and}$$

$$\left| \int f(x) + g(x) dx \right| \leq \int |f(x)| \, dx + \int |g(x)| \, dx$$

Proof $\rightarrow$ see notes

$L^1(\mathbb{R}^N)$

As consequence of Proposition 2.2 if two functions are identical almost everywhere, then they have the same integral.



Def 2.3: $L^1(\mathbb{R}^N)$ is the **vector space of equivalence classes** of functions of $\mathcal{L}^1(\mathbb{R}^N)$ identical almost everywhere.

L^1-norm: $$\|u\|_{L^1(\mathbb{R}^N)} = \|u\|_1 = \int_{\mathbb{R}^N} |u(x)|dx$$

Convergence: The sequence f_n converge to f in L^1-norm if

$$\lim_{n\to\infty} \int |f_n(x) - f(x)|dx = 0.$$

Dominated convergence in $L^1(\mathbb{R}^N)$

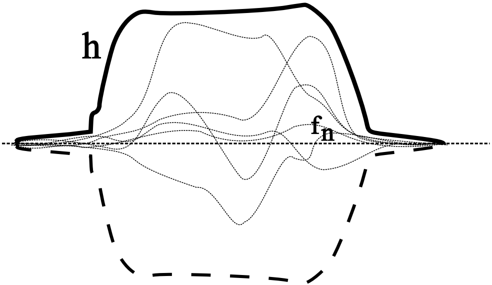
Theorem 2.1: **(Lebesgue's, dominated convergence)**

• Let f_n, sequence of functions that converge (pointwise) a.e. to f

• Suppose exists a positive and integrable function h **(chapeau)** that $\forall n \quad |f_n(x)| \leq h(x),\ a.e.$

Then $\underbrace{\int |f_n(x) - f(x)| dx \to 0}_{\text{Def: } f_n \xrightarrow{L^1} f}$ and $\int f_n(x) dx \to \int f(x) dx$

Proof → blackboard



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Proof → blackboard

Attention! $f_n \xrightarrow{L^1} f \not\Rightarrow$ pointwise convergence of $f_n(x)$ to $f(x)$ **but**

Theorem 2.2: **(Reverse of Lebesgue's Thm)**

If $f_n \xrightarrow{L^1} f$, then exists a sub sequence f_{n_k} which converges a.e. to f, and exists h integrable s.t. $\forall k\, |f_{n_k}(x)| \leq h(x)$ a.e.

Proof → Ex.17

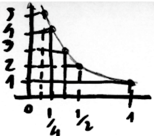
Remind: Lebesgue's, dominated convergence Thm.

- f_n: sequence converge (pointwise) a.e. to f
- exists h positive, integrable function s.t. $\forall n \quad |f_n(x)| \leq h(x)$, *a.e.*

$$\text{Then} \quad f_n \xrightarrow{L^1} f \ \text{ and } \int f_n \to \int f$$

- Find f_n s.t. f_n does not converge in $L^1([0,1])$ to 0, but $f_n(x) \to 0$ a.e.

$$f_n(x) = \begin{cases} n & \text{if } 0 \leq x \leq 1/n \\ 0 & \text{else} \end{cases}$$

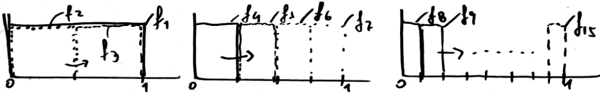


Fails because $h(x) = \frac{1}{x}$ s.t. $|f_n(x)| \leq h(x)$, is not integrable.

- Find f_n in $[0,1]$ s.t. $f_n \xrightarrow{L^1} 0$, but $f_n(x) \not\to 0$ a.e.

Si $2^{k-1} \leq n \leq 2^k$

$\Rightarrow f_n = 1$ sur $\left[\frac{n-2^{k-1}}{2^{k-1}}, \frac{n+1-2^{k-1}}{2^{k-1}}\right]$



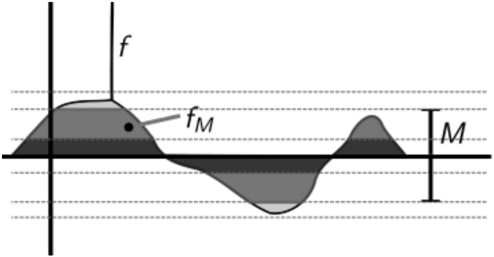
But there should be a subsequence of f_n which converges to 0 a.e.

Convergence theorems in $L^1(\mathbb{R}^N)$ Skip

Def 2.4: **Truncation:** $f^M = \max(-M, \min(M, f)),\ M > 0$.

Verifies:

1. $f^M(x) \overset{M\to\infty}{\longrightarrow} f(x)$, a.e.
2. if $f \in L^1$, then $\int |f^M - f| \overset{M\to\infty}{\longrightarrow} 0$
3. $|f^M - g^M| \leq |f - g| \quad \forall f, g$



Convergence theorems in $L^1(\mathbb{R}^N)$

Theorem 2.3: **(Bounded convergence)**

Let f_k be a sequence of functions such that:

- f_k are integrable
- are uniformly bounded: $|f_k(x)| \leq M \; \forall k, \forall x$
- Converge in measure *(en probabilité)* to f on a bounded set C

$(\forall \varepsilon > 0 \quad \lim_{k \to \infty} mes(x \in C : |f_k(x) - f(x)| > \varepsilon) = 0 \;)$

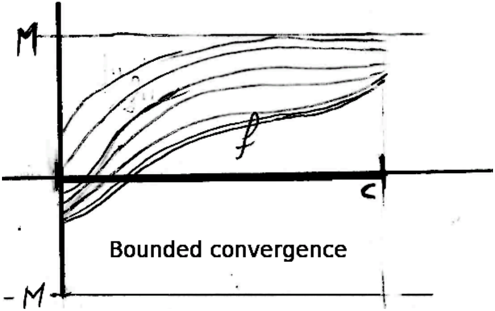
Convergence in measure is weaker than convergence a.e.

$$\text{Then} \qquad \lim_{k \to \infty} \int_C |f_k - f| = 0$$

Proof → see notes

Convergence theorems in $L^1(\mathbb{R}^N)$

Bounded convergence Example:



Counter-Examples:

• $f_n = \mathcal{X}_{[n,\infty]}$ does not converge in measure and C is not compact.

• $f_n = n\mathcal{X}_{[0,1/n]}$ converge in measure in $[0,1]$ but is not bounded.

Derivation under the sum

Theorem 2.4: **Derivation under the sum.**

Let $I \subset \mathbb{R}$, $A \subset \mathbb{R}^N$ and f defined on $A \times I$, s.t.:

$a)$ for all $\lambda \in I$ the function $x \mapsto f(x, \lambda)$ is integrable in A

$b)$ $\frac{\partial f}{\partial \lambda}(x, \lambda)$ is defined for all point of $A \times I$

$c)$ exists $h(x)$ positive, integrable over A s.t. $\left|\frac{\partial f}{\partial \lambda}(x, \lambda)\right| \leq h(x) \, \forall x, \lambda$

Then

$$F(\lambda) = \int_A f(x, \lambda) dx \quad \text{is derivable in } I$$

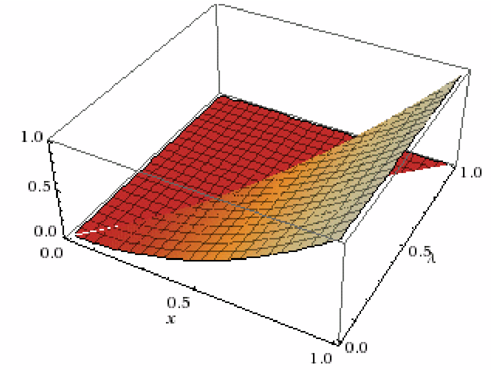
$$\text{and} \quad F'(\lambda) = \int_A \frac{\partial f}{\partial \lambda}(x, \lambda) dx$$

Proof $\rightarrow$ blackboard

Derivation under the sum

Ex 23: Let $f(x, \lambda)$ defined in $[0,1] \times [0,1]$, and ϕ continue in $[0,1]$.
$f(x, \lambda) = \phi(x)$ if $x \leq \lambda$; and $f(x, \lambda) = 0$ otherwise.
Compute $F'(\lambda)$, where $F(\lambda) = \int_0^1 f(x, \lambda) dx$.

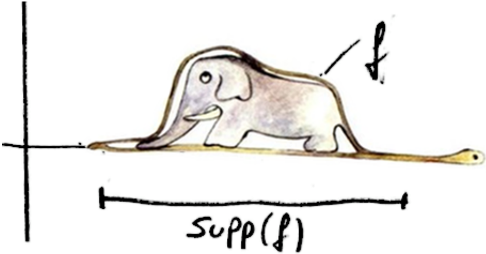
Example



Plot3D[x^2 * H[x − λ], {x, 0, 1}, {λ, 0, 1}]

Compact and simple functions (functions en escalier)

Def 2.6: For f defined over $\mathbb{R}^N$, $\underline{support(f)}$ is the complement of the largest open set O s.t. $f(x) = 0$ a.e. over O.
The $support(f)$ is closed, and when it is bounded then we say that f has **compact support**.



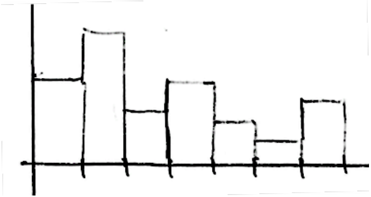
support

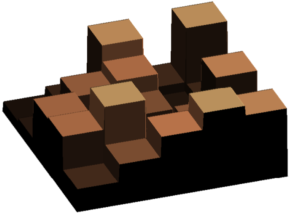
Compact and simple functions (functions en escalier)

Def 2.7: We consider a **grid (quadrillage)** of size a of $\mathbb{R}^N$, with $C = [0, a[^N$ we define the disjoint tiling $\mathbb{R}^N = \cup_{n \in \mathbb{Z}^N}(an + C)$

A simple function (en escalier)

- is constant on each hypercube of the grid of size a
- and has a compact support





simple functions

Def 2.8: **A dyadic simple function** is a simple function associated to a grid of size $a = 2^k$, $k \in \mathbb{Z}$

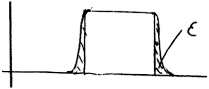
Limit of [simple, compactly supported] functions

Property 2.4: For all f, integrable in $\mathbb{R}^N$, there exists a sequence of simple functions (also dyadic) which converges to f in $L^1(\mathbb{R}^N)$.

Def 2.9: Let Ω be an open subset of $\mathbb{R}^N$, we denote $\mathcal{C}_c(\Omega)$ the set of continuous functions with compact support contained in Ω.

Using Prop.2.4 we deduce the following density property:

Corollary 2.1: For all function f, integrable in $\mathbb{R}^N$, there exists a sequence of functions $f_n \in \mathcal{C}_c(\mathbb{R}^N)$ s.t. $f_n \xrightarrow{L^1} f$.



Proof → blackboard

We say that $\mathcal{C}_c(\mathbb{R}^N)$ is dense in $L^1(\mathbb{R}^N)$.

Change of variables

Theorem 2.6: **(Change of variables)**
Let Ω_1 and Ω_2 be open subsets of $\mathbb{R}^N$, and ϕ diffeomorphism between Ω_1 and Ω_2. Let f be a function defined over Ω_2.
(a) If f is integrable and $f(x) \in \mathbb{R}^+ \cup \{+\infty\}$ then

$$\int_{\Omega_2} f(y)dy = \int_{\Omega_1} f(\phi(x))|J_\phi(x)|dx,$$

where $J_\phi(x)$ denotes the determinant of the Jacobian of ϕ at x.
(b) If f is complex valued, then it is integrable in Ω_2 **iff** $f(\phi(x))J_\phi(x)$ is integrable in Ω_1, then the previous equality holds.

Proof $\rightarrow$ see notes

Fubini-Tonelli

Property 2.5: (Fubini-Tonelli Theorem)
Let $f(x, y)$ be a function defined on $\mathbb{R}^p \times \mathbb{R}^q$.
(a) (Tonelli) If $f(x, y) \in \mathbb{R}^+ \cup \{+\infty\}$, then

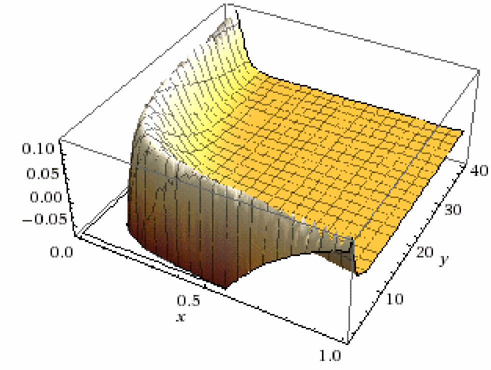
$$\int_{\mathbb{R}^{p+q}} f(x, y) dx\, dy = \int_{\mathbb{R}^p} \left(\int_{\mathbb{R}^q} f(x, y) dy \right) dx = \int_{\mathbb{R}^q} \left(\int_{\mathbb{R}^p} f(x, y) dx \right) dy$$

(b) (Fubini) If f is integrable in $\mathbb{R}^{p+q}$, then the terms of the previous equation are correctly defined and the equality holds.

Fubini-Tonelli

Ex 24.2

$$\int_0^1 \int_1^\infty e^{-xy} - 2e^{-2xy}\, dy\, dx \neq \int_1^\infty \int_0^1 e^{-xy} - 2e^{-2xy}\, dx\, dy$$

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Plot3D[(-2 + E^(x y))/E^(2 x y), {x, 0, 1}, {y, 1, 40}]
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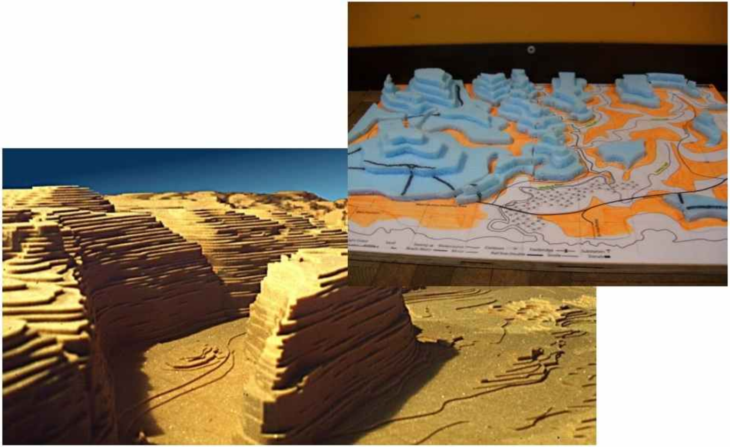
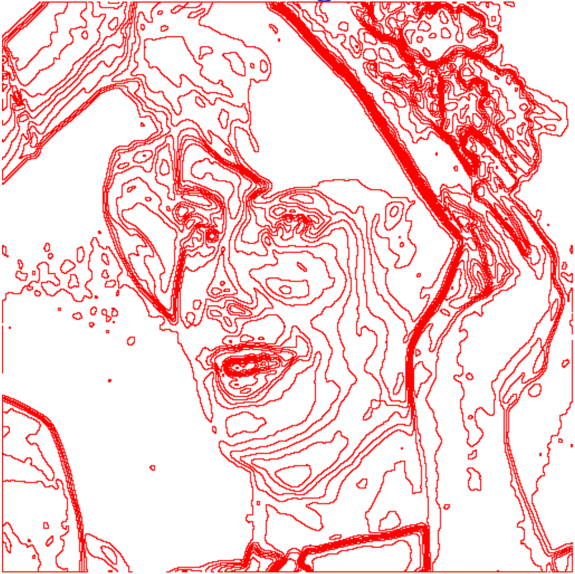
Applications: Level sets of images

An image can be seen as a function $f : \mathbb{R}^2 \to [0, \infty]$.



- It can be shown that a function is measurable if almost all its level sets are measurable. That is, the level sets that are not measurable are negligible.
- After all the Lebesgue integral is defined as sum over level sets.

Applications: Level lines of images



The graph of an image is a very irregular, mainly due to occlusions, small scale details and noise. Also the level lines are very irregular.

Summary

- Lebesgue integral, L^1 and convergence theorems
- $\mathcal{C}_c$ is dense in L^1
- Assorted integration theorems