Hilbert and Fourier analysis
C2

M. Delbracio & G. Facciolo

Previously...

- L^1 space and norm
- Convergence of sequences of functions in L^1
- Dominated convergence theorem
- $\mathcal{C}_c(\mathbb{R}^N)$ is dense in $L^1(\mathbb{R}^N)$

Today's topics

- L^p AND L^∞ SPACES
  - L^p, is a normed vector space
  - Convergence theorems in L^p
  - L^p are **Banach** spaces for $1 \leq p \leq \infty$
  - $\mathcal{C}_c(\mathbb{R}^N)$ is **dense** in $L^p(\mathbb{R}^N)$ for $1 \leq p < \infty$
- CONVOLUTION, APPROXIMATION AND REGULARIZATION
  - Convolution*
  - Approximation of the identity, and $(f * k_h) \overset{L^p}{\to} f$
  - $\mathcal{C}_c^\infty(\Omega)$ is **dense** in $L^p(\Omega)$ for $1 \leq p < \infty$

L^1, L^p, L^∞

Def 3.1: L^p **space**

- $L^p(\mathbb{R}^N)$: set of functions f defined a.e. such that

$$\int_{\mathbb{R}^N} |f(x)|^p \, dx < \infty, \quad 1 \leq p < \infty$$

- $L^\infty(\mathbb{R}^N)$: set of functions such that

$$\exists c \geq 0 : |f(x)| \leq c \text{ a.e. } x \tag{1}$$

L^p-norms
$\|f\|_{L^p} =$ (or simply $\|f\|_p$) $= \left(\int_{\mathbb{R}^N} |f(x)|^p \, dx\right)^{\frac{1}{p}}$
$\|f\|_{L^\infty} =$ supess(f): (Essential sup.) smallest c for which (1) holds

Objective: **Prove that L^p with $1 \leq p < \infty$ is a Banach space (complete, normed, vector space)**

L^p: normed vector space

To prove that L^p is a vector space we'll need:

Theorem 3.1: **(Hölder inequality)**
Let $1 \leq p, p' \leq \infty$ with $p' = \frac{p}{p-1}$ (conjugate exponents).
Then for all $f \in L^p$ and $g \in L^{p'}$, the function $fg \in L^1$ and:

$$\|fg\|_1 = \int |fg| \leq \|f\|_p \|g\|_{p'}.$$

Proof $\rightarrow$ see notes

Théorème 3.1 *(Inégalité de Hölder) Soit $1 \leq p \leq +\infty$. Pour tout $f \in L^p$ et $g \in L^{p'}$, la fonction fg est sommable et* $$\int |fg| \leq ||f||_p ||g||_{p'}.$$

Démonstration Supposons d'abord $1 < p < +\infty$. On utilise la concavité du logarithme sur $\mathbb{R}^+$. Si $a > 0$, $b > 0$, compte tenu de la relation $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\log(\frac{1}{p}a^p + \frac{1}{p'}b^{p'}) \geq \frac{1}{p}\log(a^p) + \frac{1}{p'}\log(b^{p'}) = \log(ab). \text{ D'où}$$

Young's Inequality -> $$ab \leq \frac{1}{p}a^p + \frac{1}{p'}b^{p'} \text{ et donc}$$

$$|f(x)g(x)| \leq \frac{1}{p}|f(x)|^p + \frac{1}{p'}|g(x)|^{p'}, \text{ qui donne}$$

$$\int |f(x)g(x)| \leq \frac{1}{p}||f||_{L^p}^p + \frac{1}{p'}||g||_{L^{p'}}^{p'}.$$

Pour trouver l'inégalité recherchée, on remplace le couple (f, g) par $(\lambda f, \frac{g}{\lambda})$, où $\lambda > 0$.

On choisit $\boxed{\lambda = \frac{||g||_{p'}^{\frac{1}{p}}}{||f||_p^{\frac{1}{p'}}}}$. Alors $$\int |fg| \leq (\frac{1}{p} + \frac{1}{p'})||f||_p ||g||_{p'},$$

qui est bien l'inégalité annoncée. Si $p = 1$ ou $p = +\infty$, on a $p' = +\infty$ ou $p' = 1$ et l'inégalité est immédiate en choisissant un représentant de g tel que $\sup |g| = ||g||_\infty$: on a alors $\int |fg| \leq \sup |g| \int |f| = ||g||_\infty ||f||_{L^1}$. $\circ$

$L^p(\mathbb{R}^N)$: normed vector space

Theorem 3.2: $L^p(\mathbb{R}^N)$ **(with** $1 \le p \le \infty$**) is a vector space and** $\|f\|p$ **is a norm**

Points to prove:

1. $\|\lambda f\|_p = |\lambda| \|f\|_p$: by Definition of L^p
2. $\|f\|_p = 0 \Rightarrow f = 0$ a.e.: easy
3. $\|f+g\|_p \le \|f\|_p + \|g\|_p$: use Hölder.

Proof of 3.

$$\underline{\|f+g\|_p^p} = \int |f+g|^p = \int |f+g|^{p-1} |f+g| \le \quad \text{(triangle)}$$

$$\int \underbrace{|f+g|^{p-1}}\underbrace{|f|} + \int \underbrace{|f+g|^{p-1}}\underbrace{|g|} \le \quad \text{(Hölder x2)}$$

$$\left\| \, |f+g|^{p-1} \right\|_{p'} \|f\|_p + \left\| \, |f+g|^{p-1} \right\|_{p'} \|g\|_p = \quad \text{(comput...)}$$

$$\underline{\|f+g\|_p^{p-1}} (\|f\|_p + \|g\|_p)$$

Convergence in $L^p(\mathbb{R}^N)$

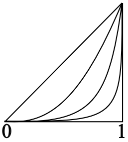
Proposition 3.1: **(Dominated convergence in L^p, $1 \le p < \infty$)**
Let $f_n(x) \to f(x)$ pointwise a.e., and exists $g \in L^p$ **(chapeau)** such that $|f_n(x)| \le g(x)$ *a.e.* for all n. Then $f_n \xrightarrow{L^p} f$.

Proof →

Démonstration On pose $h_n = |f_n - f|^p$. Alors $|h_n| \le (|f_n| + |f|)^p \le 2^p |g|^p$ p.p. car $|f_n| \le g$ p.p. et, en passant à la limite simple, $|f| \le g$ presque partout. On peut donc appliquer le théorème de la convergence dominée à h_n et conclure que $h_n \to 0$. ○

Exercise 3. Dominated convergence counterexample for $L^\infty([0,1])$.
Let : $f_n(x) = x^n$ over $[0,1]$:

$$f_n \in L^\infty([0,1]) \ \checkmark \quad ,$$
$$|f_n(x)| \le 1 \ \checkmark \quad ,$$
$$f_n(x) \to 0 \text{ a.e. } x \in [0,1] \ \checkmark$$



But $\|f_n - 0\|_\infty = \text{supess}\,|f_n| = 1 \ \forall n.$

... we also have Lemma 3.1: **(Fatou in L^p for $1 \le p \le \infty$).**

$$\text{If } \begin{cases} \bullet\ f_n(x) \to f(x) \text{ a.e.} \\ \bullet\ \|f_n\|_p \le c \end{cases}, \quad \text{then} \quad \|f\|_p \le \liminf_n \|f_n\|_p$$

Proof → see notes

$L^p(\mathbb{R}^N)$ is Banach

Theorem 3.3: **(Fischer-Riesz)**
$L^p(\mathbb{R}^N)$ is **complete** for $1 \leq p \leq \infty$ (and therefore **is Banach**).
"Complete: if every Cauchy seq. in L^p has a limit that is also in L^p."
Proof → blackboard

Examples of Non Banach spaces

1. $\mathbb{Q}$ with standard metric:
   seqs. of rationals can converge to elements of $\mathbb{R}$
2. Open intervals of $\mathbb{R}$ with standard metric:
   $X = (0, 1] \quad \frac{1}{n} \to 0 \notin X$
3. $\mathcal{C}_c$ with L^1 metric:
   seq. $f_n(x) = x^n \in \mathcal{C}_c([0, 1])$ converges to an $f(x) \notin \mathcal{C}_c([0, 1])$

Théorème 3.3 *Les espaces $L^p(\mathbb{R}^N)$, $(1 \le p \le +\infty)$ sont des espaces de Banach.*

Démonstration Soit h_k une suite de Cauchy dans L^p. Si $p = +\infty$, $h_k(x)$ est de Cauchy pour presque tout x et converge uniformément vers une fonction $h(x)$. Il est alors facile de vérifier que $h(x)$ est essentiellement bornée et que la convergence de h_k vers h est uniforme. Soit maintenant $1 < p < \infty$. Une récurrence facile (exercice !) montre que l'on peut extraire une suite de la suite de Cauchy h_k, que nous noterons par commodité $f_n = h_{k_n}$, telle que

$$q, r \ge n \Rightarrow ||f_q - f_r||_{L^p} \le \frac{1}{2^n}.$$

On pose $\boxed{g_n(x) = \sum_1^n |f_{k+1}(x) - f_k(x)|}$ La suite de fonctions $g_n(x)$ est croissante. Notons $g(x)$ sa limite. Par l'inégalité triangulaire,

$$\left(\int |g_n|^p\right)^{\frac{1}{p}} = ||\sum_{k=1}^{n} |f_{k+} - f_k|\,||_p \le \sum_{k=1}^{n} ||f_{k+1} - f_k||_p \le 1.$$

En appliquant à $|g_n|^p$ le théorème de convergence monotone, on obtient donc

$$\int |g(x)|^p = \lim_n \int |g_n|^p \le 1.$$

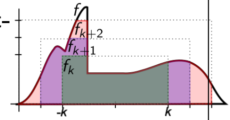
Il en résulte que $g(x)$ est presque partout finie, appartient à L^p et que $g_n(x)$ tend presque partout vers $g(x)$. Comme (pour $n \ge m$), $\sum_m^n |f_{k+1} - f_k|(x) \le g_n(x) - g_m(x)$, la série $\sum_1^\infty (f_{k+1} - f_k)$ converge également presque partout. Donc $f_n(x) - f_1(x)$ converge presque partout vers une limite que l'on appelle $f(x) - f_1(x)$. De plus, $|f_n(x) - f(x)|^p \le (\sum_n^\infty |f_{k+1} - f_k(x)|)^p \le g(x)^p$ qui est chapeau intégrable. Donc $f_n - f$ converge vers 0 dans L^p. Rappelons que $f_n = h_{n_k}$ est une sous-suite extraite de la suite de Cauchy h_k. Mais si une sous-suite d'une suite de Cauchy converge, toute la suite converge.

$\mathcal{C}_c$ dense in L^p for $1 \leq p < \infty$

Theorem 3.4: $\mathcal{C}_c(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$ for $1 \leq p < \infty$

Proof $\rightarrow$ see notes

Sketch



1. For all $f \in L^p$ we define a seq. of **truncated** functions f_k: bounded and with compact support.

   By dominated convergence $f_k \xrightarrow{L^p} f$

2. Since f_k is bounded and compactly supported **then** $f_k \in L^1$, and by density of C_c in L^1 there exists $g_n \in \mathcal{C}_c$ such that $g_n \xrightarrow{L^1} f_k$.

3. The reverse Lebesgue Thm assures that a subsequence of g_n converges pointwise to f_k. Since $(g_n)_n$ and f_k are bounded and compactly supported, it is easy to find a dominating function in L^p to apply dominated convergence to get: $g_n \xrightarrow{L^p} f_k$

Exercise 7.

1. Is $\mathcal{C}_c(\mathbb{R})$ dense in $L^\infty(\mathbb{R})$?
2. Is $\mathcal{C}_c(\mathbb{R})$ dense in $L^1([0,1])$?

Convolution, approximation and regularization

We've seen that $\mathcal{C}_c$ is dense in L^p. This implies that we can approximate an L^p function as much as we want using continuous functions. However these functions may not be smooth (discontinuous derivatives).

In this part we'll construct sequences of infinitely differentiable functions for approximating L^p functions.

Convolution: The convolution of $f, g \in L^1(\mathbb{R}^N)$ is defined as

$$(f * g)(x) = \int f(x-y)g(y)dy = \int g(x-y)f(y)dy$$

Convolution, approximation and regularization

$$\boxed{\text{Convolution: } (f * g)(x) = \int f(x-y)g(y)dy = \int g(x-y)f(y)dy}$$

Theorem 3.6:

Let $f, g \in L^1(\mathbb{R}^N)$. Then $f(x-y)g(y)$ is integrable for a.e. x, the convolved function $f * g$ is in L^1 and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

Proof → see notes

Démonstration On montre d'abord en appliquant le théorème de Fubini-Tonelli et le théorème de changement de variable que la fonction $|f(x-y)g(y)|$ est sommable. En effet,

$$\int\int |f(x-y)g(y)|dxdy = \int |g(y)|(\int |f(x-y)|dx)dy =$$

$$\int |g(y)|(\int |f(x)|dx)dy = ||f||_{L^1}||g||_{L^1}.$$

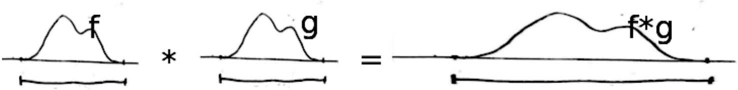
La fonction $f(x-y)g(y)$ est donc sommable sur $\mathbb{R}^{2N}$ et le théorème de Fubini assure que la fonction $y \to f(x-y)g(y)$ est sommable pour presque tout x et que la fonction alors définie presque partout par $f * g(x) = \int f(x-y)g(y)dy$ est elle-même sommable sur $\mathbb{R}^N$. On a finalement

$$||f * g||_{L^1} = \int |f * g|(x)dx \leq \int |g(y)|(\int |f(x-y)|dx)dy = ||f||_{L^1}||g||_{L^1}.$$

Support and convolution

Def 2.6: The support of f, $Supp(f)$ is the complement of the largest open set O s.t. $f(x) = 0$ a.e. over O.

Proposition 3.2: Let f, g be such that $(f * g)(x)$ is defined a.e..
Then $\quad \mathrm{Supp}(f * g) \subset \overline{\mathrm{Supp}(f) + \mathrm{Supp}(g)}$.

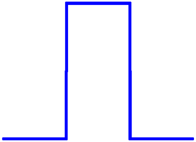


Proof → see notes

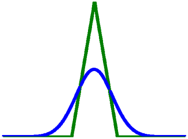
Regularization by convolution

Theorem 3.7: **(Regularization by convolution)**
Let $f \in L^1(\mathbb{R}^N)$, $g \in \mathcal{C}(\mathbb{R}^N)$ bounded ($\|g\|_\infty < \infty$) and with partial derivatives $\frac{\partial g}{\partial x_i}$ continues up to order k. Then the convolved function $(f * g)(x) = \int g(x-y) f(y) dy$ **is defined for all x**, $f * g$ is **continue and bounded**, and the partial derivatives $\frac{\partial f * g}{\partial x_i}$ **are continues up to order** k.

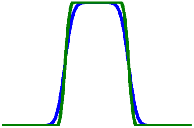
Proof → blackboard



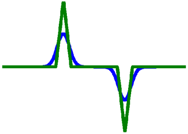
f



g



$f * g$



$\partial_x (f * g)$

Démonstration On remarque d'abord que la fonction $y \to g(x-y)f(y)$ est som-

L^1_{loc} SKIP

Def 3.2: We denote $L^1_{loc}(\mathbb{R}^N)$ the space of functions f such that: **for all bounded set B** $\subset \mathbb{R}^N$, $f \in L^1(B)$.

We say that $f_n \xrightarrow{L^1_{loc}} f$ if $f_n \xrightarrow{L^1(B)} f$ for all bounded set B.

Exercise 8. $[\mathcal{C}^0(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N), 1 \leq p \leq \infty] \subset L^1_{loc}(\mathbb{R}^N)$.

Corollary 3.2: If $f \in L^1_{loc}$ and $g \in \mathcal{C}^k_c$, then $f * g \in \mathcal{C}^k_c$.

Proof $\rightarrow$ Ex. 9

L^2 SKIP

Theorem 3.8:
Let $f \in L^1(\mathbb{R}^N)$ and $g \in L^2(\mathbb{R}^N)$, then $f * g(x)$

- is defined almost everywhere,
- is square-integrable,
- and verifies $\|f * g\|_2 \leq \|f\|_1 \|g\|_2$.

Let $f, g \in L^2(\mathbb{R}^N)$, then $f * g$

- is defined everywhere,
- is continuous,
- tends towards 0 at the infinity,
- and verifies $\|f * g\|_\infty \leq \|f\|_2 \|g\|_2$.

Fundamental theorem of signal processing

Theorem 3.9: (Universality of the convolution)
Let $T : L^2(\mathbb{R}^N) \to C_b(\mathbb{R}^N)$ be a continuous linear operator which is translation invariant.

Then there exist a unique $g \in L^2(\mathbb{R}^N)$ such that $T(f) = g * f$.

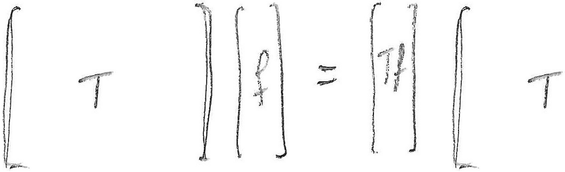
Proof → see notes

Missing definitions:

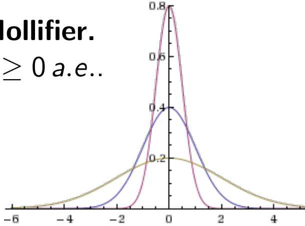
- C_b set of continuous and bounded funct., equipped with the L^∞-norm
- Translation of a function by x: $(\tau_x f)(y) = f(y - x)$
- Translation invariant operator: $T(\tau_x f) = \tau_x T(f)$

Discrete intuition:

- Linear Operator T
- Translation invariant
- Kernel g (row of T)



Approximations to the identity



Def 3.4: **Approximation of the identity or Mollifier.**
Let $k \in L^1(\mathbb{R}^N)$ so that $\int k(x)dx = 1$ and $k(x) \geq 0$ *a.e.*.
For all $h > 0$ we define: $\boxed{k_h(x) = h^{-N} k\left(\frac{x}{h}\right).}$
We call k_h approximation of the identity.

Theorem 3.5: **("Average continuity" in L^p)**
For all $f \in L^p(\mathbb{R}^N), 1 \leq p < \infty$ we have
$\lim_{y \to 0} \int_{\mathbb{R}^N} |f(x+y) - f(x)|^p dx = 0.$

Proof → see notes

Proposition 3.3: If f is uniformly continuous and bounded in R^N, then $f * k_h$ converges uniformly to f.

Proof → blackboard

Définition 3.3 *Soit $k \in L^1(\mathbb{R}^N)$ telle que $\int k(x)dx = 1$ et $k(x) \geq 0$ presque partout. On pose pour tout $h > 0$,*

$$k_h(x) = h^{-N} k\left(\frac{x}{h}\right).$$

On appelle un tel système de fonctions k_h, $h > 0$, une "approximation de l'identité".

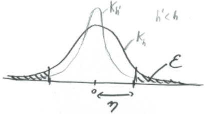
Proposition 3.3 *Si f est uniformément continue et bornée sur $\mathbb{R}^N$, alors $f * k_h$ converge vers f uniformément.*

Démonstration On remarque d'abord que

$$\int k_h(x)dx = \int k(x)dx = 1, \tag{3.7}$$

$$k_h * f(x) - f(x) = \int (k_h(y)(f(x-y) - f(x))dy, \tag{3.8}$$

et que (une conséquence du théorème de Lebesgue), pour η fixé et h assez petit,



$$\boxed{\int_{|y|\geq\eta} |k_h(y)|dy \leq \varepsilon.} \tag{3.9}$$

Donc

$$|(k_h * f)(x) - f(x)| = \left|\int k_h(y)(f(x-y) - f(x))dy\right|$$

$$\leq 2\sup|f| \int_{|y|\geq\eta} |k_h(y)|dy + \sup_{|y|\leq\eta} |f(x-y) - f(x)| \int |k_h(y)|dy \leq 2\varepsilon$$

pourvu que l'on fixe premièrement η assez petit pour que le deuxième terme soit plus petit que ε, puis h assez petit pour que le premier soit aussi plus petit que ε. ◦

Approximations to the identity

Theorem 3.10: Let k_h be a mollifier.
Then for all $f \in L^p(\mathbb{R}^N)$, $1 \leq p < \infty$, the functions $f * k_h \xrightarrow{L^p} f$.

Proof $\rightarrow$ blackboard

Théorème 3.10 *Soit k_h une approximation de l'identité. Alors pour toute fonction $f \in L^p(\mathbb{R}^N)$, $1 \leq p < \infty$, les fonctions $f * k_h$ convergent vers f dans L^p.*

Démonstration Comme $f \in L^p$, on peut choisir par le théorème 3.5 η assez petit pour que

$$\underline{|y| \leq \eta \Rightarrow \int |f(x-y) - f(x)|^p dx \leq \varepsilon.} \tag{3.10}$$

L'inégalité de Hölder et le fait que $k_h \geq 0$, $\int k_h = 1$ impliquent que pour toute fonction f dans L^p, on ait

$$\left(\int k_h(x)|f(x)|dx\right)^p \leq \int k_h(x)|f(x)|^p dx. \tag{3.11}$$

En effet,

$$\left(\int k_h|f|\right)^p = \left(\int k_h^{\frac{1}{p}} k_h^{\frac{1}{p'}} |f|\right)^p \leq \left(\int k_h|f|^p\right)\left(\int k_h\right)^{\frac{p}{p'}}.$$

En utilisant les relations (3.7)-(3.11) et le théorème de Fubini,

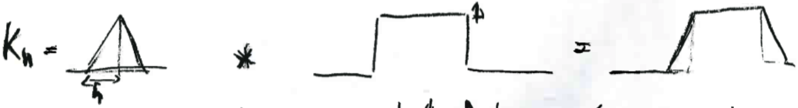
$$\int |k_h * f(x) - f(x)|^p dx = \int \left|\int (k_h(y)(f(x-y) - f(x))dy\right|^p dx \leq \int \int k_h(y)|f(x-y) - f(x))|^p dy dx \leq$$

$$2^p \int |f(x)|^p dx \underline{\int_{|y|\geq\eta} |k_h(y)|dy} + \int_{|y|\leq\eta} |k_h(y)|dy \underline{\int |f(x-y) - f(x)|^p dx} \leq$$

$$2^p ||f||^p_{L^p}\varepsilon + \int |k_h(y)|\varepsilon = (2^p||f||^p_{L^p} + ||k||_{L^1})\varepsilon.$$

Approximations to the identity

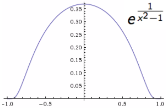
Ex. 11 Case $p = \infty$ in the previous theorem.
Show an example where $f * k_h$ does not converge in L^∞ to f.

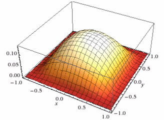


$\lim_{h \to 0} \operatorname{esssup} |H - K_h * H| = \operatorname{supess} | \quad | = 1 \quad \forall h \Rightarrow$ FAIL!

Density of $\mathcal{C}_c^\infty$ in L^p

Lemma 3.2: Existence of $\mathcal{C}_c^\infty$ functions. Ex: $e^{\frac{1}{x^2-1}}, x \in [-1, 1]$





Proposition 3.4: $\mathcal{C}_c^\infty(\Omega)$ is dense in $L^p(\Omega)$

Proof → blackboard

Proposition 3.4 *Soit Ω un ouvert de $\mathbb{R}^N$. Alors $\mathcal{C}_c^\infty(\Omega)$ est dense dans $L^p(\Omega)$ pour tout $1 \leq p < \infty$.*

Démonstration On considère les voisinages intérieurs bornés de Ω, $\Omega_r = \{x, B(x,r) \subset \Omega\} \cap B(0, \frac{1}{r})$. Comme $\Omega = \cup_{r>0} \Omega_r$, on a par le théorème de Lebesgue

$$\forall \varepsilon \exists r, \int_\Omega |f(x) - f(x) 1\!\!1_{\Omega_r}|^p dx < \varepsilon.$$

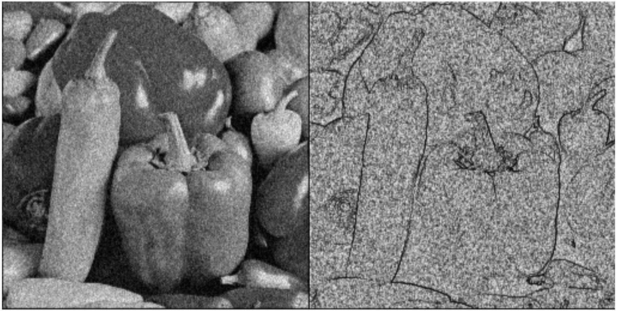
On choisit alors $r' < r$ tel que

$$||(f 1\!\!1_{\Omega_r}) * k_{h_{r'}} - f 1\!\!1_{\Omega_r}||_{L^p(\Omega)} < \varepsilon.$$

On obtient donc $\boxed{||f - (f 1\!\!1_{\Omega_r}) * k_{h_{r'}}||_{L^p(\Omega)} < 2\varepsilon.}$ Comme $r' < r$, le lemme 3.2 et le théorème 3.7 impliquent que $(f 1\!\!1_{\Omega_r}) * h_{r'} \in \mathcal{C}_c^\infty(\Omega)$. ◦

Regularization by convolution for images

Digital images usually contain noise. Noise makes the image very irregular, difficulting the computation of image derivatives.
The image derivatives are relevant because places with **high derivative** often coincide with **object boundaries**.
But computing the derivatives $Df = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ of a noisy image produces poor results.



Left image f, right $|Df|$

Regularization by convolution for images

Convolving f with a smooth function $k \in C^{\infty}$, assures that $f * k$ is smooth, and also its derivatives.