Hilbert and Fourier analysis
C5

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- We've seen that the **hermitian Hilbert space** $L^2(-\pi, \pi)$ admits a base of sinusoids

$$\frac{1}{\sqrt{2\pi}} \left(e^{inx}\right)_{n \in \mathbb{Z}}$$

that we call Fourier base.

- Then, for any $f \in L^2(-\pi, \pi)$ we can write its **Fourier series**

$$f(x) = \sum_{n \in \mathbb{Z}} c_n(f) e^{inx} \quad \text{(in the sense of the } L^2 \text{ norm)}$$

where

$$c_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

are the **Fourier coefficients**.

Today's topics

- Fourier Series
  - 2D Fourier bases
  - Decreasing of Fourier coefficients and application to JPEG compression
  - Gibbs Phenomenon
  - Summary

2D Fourier base

Now, we describe the 2D Fourier bases.

Notation: Let's write the vector components as $x = (x_1, x_2) \in \mathbb{R}^2$, $k = (k_1, k_2) \in \mathbb{R}^2$ and denote its product: $x\,k = x_1 k_1 + x_2 k_2$.

Definition: **Separable functions.** A function of 2 independent variables is said to be separable if it can be expressed as a product of 2 functions, each of them depending on only one variable.

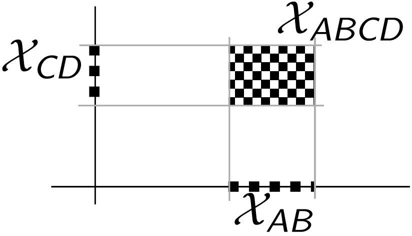
$$w(x_1, x_2) = u(x_1) v(x_2)$$

2D Fourier base

Lemma 5.2:
The **separable functions** of the form $w(x) = u(x_1)v(x_2)$ with $u, v \in L^2(0, 2\pi)$ **form a system which is total** for $L^2([0, 2\pi]^2)$.

Proof → blackboard

Proof:
The characteristic functions of rectangles are separable, and they form a total system of $L^2([0, 2\pi]^2)$ because by Property 2.4 they are dense in $L^1([0, 2\pi]^2)$.



2D Fourier base

Lemma 5.3:

If $u_k(x) \to u(x)$ and $v_l(x) \to v(x)$ in $L^2(0, 2\pi)$, then $u_k(x_1)v_l(x_2) \to u(x_1)v(x_2)$ in $L^2([0, 2\pi]^2)$ with l, $k \to \infty$.

Proof $\to$ blackboard

Proof:

- Observe that the product of $u, v \in L(0, 2\pi)$ is in $L^2([0, 2\pi]^2)$ since, by applying Fubini's theorem we can derive:

$$\|u(x_1)v(x_2)\|_{L^2([0,\ 2\pi]^2)} = \|u(x_1)\|_{L(0,\ 2\pi)}\|v(x_2)\|_{L(0,\ 2\pi)}$$

- The L^2-norm of the difference

$$\|u_k(x_1)\overbrace{v_l(x_2) - u(x_1)}^{\pm u(x_1)v_l(x_2)}v(x_2)\|_{L^2([0,\ 2\pi]^2)} \leq$$

$$\text{(triangle inequality)} \qquad \leq \|(u_k - u)v_l\|_{L^2([0,\ 2\pi]^2)} + \|u(v_l - v)\|_{L^2([0,\ 2\pi]^2)} =$$

$$= \underbrace{\|u_k - u\|_{L^2(0,2\pi)}}_{\to 0}\|v_l\|_{L^2(0,2\pi)} + \|u\|_{L^2(0,2\pi)}\underbrace{\|v_l - v\|_{L^2(0,2\pi)}}_{\to 0} \to 0$$

goes to 0 when $k,\ l \to +\infty$.

Theorem 5.3:

The functions $e_k(x) = \frac{1}{2\pi} e^{ikx}$, $\quad k \in \mathbb{Z}^2$ form a Hilbert base of $L^2([0,2\pi]^2)$, then for any function $u \in L^2([0,2\pi]^2)$ we have:

$$u(x) = \sum_{k \in \mathbb{Z}^2} c_k(u) e^{ikx} \quad \text{with} \quad c_k(u) = \frac{1}{(2\pi)^2} \int_{[0,2\pi]^2} u(x) e^{-ikx} dx, \qquad (5.2)$$

and the series converge in in the sense of the L^2-norm.

Démonstration On vérifie facilement que e_k est un système orthonormé. Pour montrer qu'il est total, il suffit de montrer, par le lemme 5.2, que les e_k engendrent les fonctions séparables. Mais si $w(x) = u(x_1)v(x_2) \in L^2([0,2\pi])^2$ est une telle fonction, par une application directe du théorème de Fubini, $u(x_1)$ et $v(x_2)$ sont dans $L^2(0,2\pi)$. Les fonctions u et v sont donc sommes au sens L^2 de leurs séries de Fourier :

$$u(x_1) = \sum_{k_1 \in \mathbb{Z}} c_{k_1} e^{ik_1 x_1}, \quad c_{k_1} = \frac{1}{2\pi} \int_{[0,2\pi]} u(x_1) e^{-ik_1 x_1};$$

$$v(x_2) = \sum_{k_2 \in \mathbb{Z}} c_{k_2} e^{ik_2 x_2}, \quad c_{k_2} = \frac{1}{2\pi} \int_{[0,2\pi]} v(x_1) e^{-ik_2 x_2}.$$

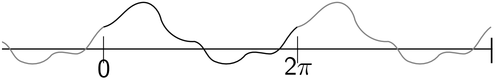
En appliquant le lemme 5.3 à $u_N(x_1) = \sum_{-N}^{N} c_{k_1}(u) e^{ik_1 x_1}$ et $v_M(x_2) = \sum_{-M}^{M} c_{k_2}(v) e^{ik_2 x_2}$ qui convergent respectivement vers $u(x_1)$ et $v(x_2)$ dans $L^2([0,2\pi])$, on obtient une série double convergente dans $L^2([0,2\pi]^2)$. On obtient donc (5.2) dans le cas d'une fonction séparable $w(x) = u(x_1)v(x_2)$ avec $c_k(w) = c_{k_1}(u)c_{k_2}(v)$. Il en résulte que le système $(e_k)_{k\in\mathbb{Z}^2}$ est une base hilbertienne de $L^2([0,2\pi]^2)$ et (5.2) est donc valide. ◦

Decreasing of Fourier coefficients and compression

- The Fourier coefficients of a 2π-periodic function $f \in C^p$ are:

$$c_n(f) = \frac{1}{2\pi}\int_0^{2\pi} e^{-inx} f(x)dx = \frac{1}{2\pi}\frac{1}{(in)^p}\int_0^{2\pi} e^{-inx} f^{(p)}(x)dx,$$

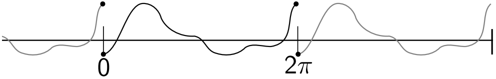
they decrease faster for higher p, that is for more regular functions.



- If f is C^1 over $[0, 2\pi]$ but **not** 2π-**periodic** (so that it presents a discontinuity at 0) then after integrating by parts the coefficients become:

$$c_n(f) = \frac{1}{2\pi}\frac{1}{in}\underbrace{\int_0^{2\pi} e^{-inx} f'(x)dx}_{(\text{R.L.})\overset{n\to\infty}{\longrightarrow}0} + \underbrace{\frac{f(0^+) - f(2\pi^-)}{2\pi(in)}}_{O(\frac{1}{n})}.$$

The first term is $o(\frac{1}{n})$, but $c_n(f)$ is still $O(\frac{1}{n})$ because of the jump at 0.



Decreasing of Fourier coefficients and compression

A decay of the Fourier coefficients of $O(\frac{1}{n})$ implies that 1000 terms of the sum are needed to attain an approximation error of 10^{-3}:

$$|S_N f(x) - f(x)|^2 = \sum_{n \geq N} |c_n(f)|^2 \leq \sum_{n \geq N} \left(\frac{1}{n}\right)^2 = O\left(\frac{1}{n}\right).$$

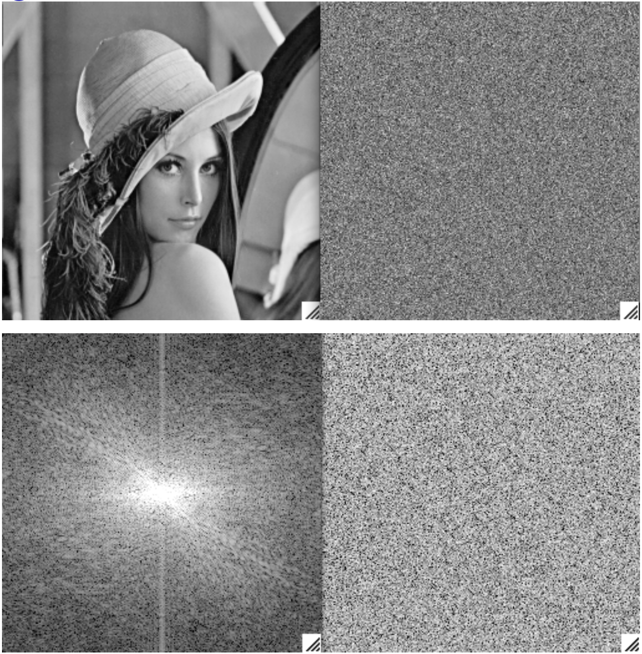
The same happens in 2D.

A slow decay of the coefficients impacts in the compression rates of the signal. An alternative to handle the discontinuities due to the periodization, is to use the cosine transform.

$$c_n(f) = \frac{1}{\pi} \int_0^{2\pi} \cos(nx) f(x) dx$$

As we've seen the cosine transform amounts to symmetrize the signal, avoiding the jumps at the boundaries due to the periodization.

Decreasing of Fourier coefficients



Application: JPEG (monochromatic)

Basic JPEG

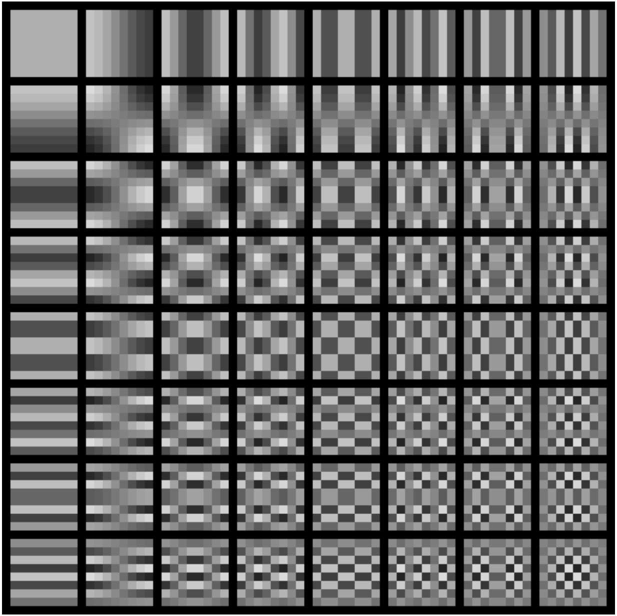
1. Convert color image to luminance (Y)
2. Split the image in blocks of size 8x8
3. For each block:
    - compute its 2D-DCT transform
    - quantize the 2D-DCT coefficients
4. Differential coding of DC coefficients
5. Zigzag scan, run-length and Huffman coding of AC coefficients

Discrete Cosine Transform (DCT)

The DCT takes a vector $x \in R^N$ and returns $X \in R^N$ as given by:

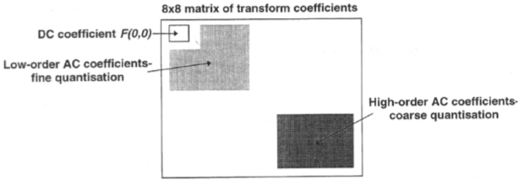
$$X_k = \sum_{n=0}^{N-1} x_n \cos\left[\frac{\pi}{N}\left(n + \frac{1}{2}\right)k\right] \qquad k = 0, \ldots, N-1$$

2D-DCT base vectors (8×8)



JPEG - quantization

✔ *Linear* quantisation with variable step size adapted to spectral order:



✔ Reconstruction level *index* transmitted: rounding to nearest integer of division of each coefficient by the step size $Q(u,v)$:

$$\boxed{F_Q(u,v) = \left\lfloor \frac{F(u,v)}{Q(u,v)} \right\rfloor}$$

✔ Inverse quantisation: multiplication of received index $F_Q(u,v)$ by the corresponding step size $Q(u,v)$:

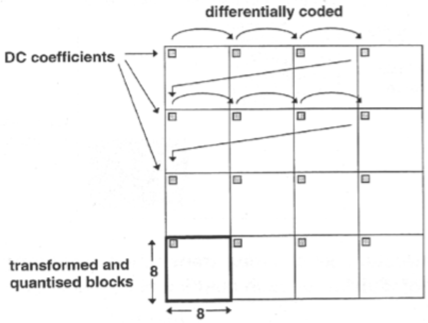
$$\boxed{F_Q^{-1}(u,v) = F_Q(u,v) \cdot Q(u,v)}$$

✔ Quantisation step size matrix $Q(u,v)$ for luminance:

16	11	10	16	24	40	51	61
12	12	14	19	26	58	60	55
14	13	16	24	40	57	69	56
14	17	22	29	51	87	80	62
18	22	37	56	68	109	103	77
24	35	55	64	81	104	113	92
49	64	78	87	103	121	120	101
72	92	95	98	112	100	103	99

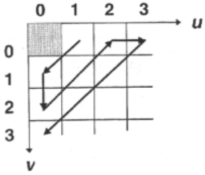
JPEG - coding of DC coefficients

✔ DC coefficients $F_Q(0,0)$ are differentially encoded, separately from the AC coefficients:



JPEG - coding of AC coefficients

- **A *zig-zag* scan converts the 8x8 2-D array $F_Q(u,v)$ of quantised AC DCT coefficients to a 1-D sequence:**



so that the coefficients are scanned according to the order: $F_Q(0,1) \rightarrow F_Q(1,0) \rightarrow F_Q(2,0) \rightarrow F_Q(1,1) \rightarrow F_Q(0,2) \rightarrow (F_Q\ 0,3) \rightarrow F_Q(1,2) \rightarrow F_Q(2,1) \rightarrow F_Q(3,0)$ and so on.

- **Due to quantisation most coefficients in the zig-zag scan sequence (predominantly those of high spectral order) will be zero.**

JPEG - example

✔ **8x8 block from source image "LENA":**

139	144	149	153	155	155	155	155
144	151	153	156	159	156	156	156
150	155	160	163	158	156	156	156
159	161	162	160	160	159	159	159
159	160	161	162	162	155	155	155
161	161	161	161	160	157	157	157
162	162	161	163	162	157	157	157
162	162	161	161	163	158	158	158

✔ **DCT-transformed block:**

1260	−1	−12	−5	2	−2	−3	1
−23	−17	−6	−3	−3	0	0	−1
−11	−9	−2	2	0	−1	−1	0
−7	−2	0	1	1	0	0	0
−1	−1	1	2	0	−1	1	1
2	0	2	0	−1	1	1	−1
−1	0	0	−1	0	2	1	−1
−3	2	−4	−2	2	1	−1	0

✔ **Psychovisually quantised block:**

79	0	−1	0	0	0	0	0
−2	−1	0	0	0	0	0	0
−1	−1	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

JPEG - example

- ✔ **Re-ordering of coefficients:**

79	0	-2	-1	-1	-1	0	0	-1	EOB

zig-zag scanned AC coefficients

DC coefficient (to be coded differentially)

- ✔ **Computation of symbols for AC coefficients:**

symbol-1 ZERO-RUN-LENGTH, SIZE	symbol-2 AMPLITUDE
1, 2	-2
0, 1	-1
0, 1	-1
0, 1	-1
2, 1	-1

- ✔ **Codeword assignment :**

symbol-1	symbol-2
111001	00
00	0
00	0
00	0
11011	0

- ✔ **Coded bit-stream:**

DC	111001	00	00	0	00	0	00	0	11011	0	EOB

- ✔ **Compression ratio:** ***8 pels* x *8 pels* x *8 bpp / 35 bits* ≈ *15:1***
  or ***0.55 bpp*** (including 8 bits for DC and 4 bits for EOB)

JPEG - example

✔ **Inverse quantisation and DCT:**

144	146	149	152	154	156	156	156
148	150	152	154	156	156	156	156
155	156	157	158	158	157	156	155
160	161	161	162	161	159	157	155
163	163	164	163	162	160	158	156
163	163	164	164	162	160	158	157
160	161	162	162	162	161	159	158
158	159	161	161	162	161	159	158

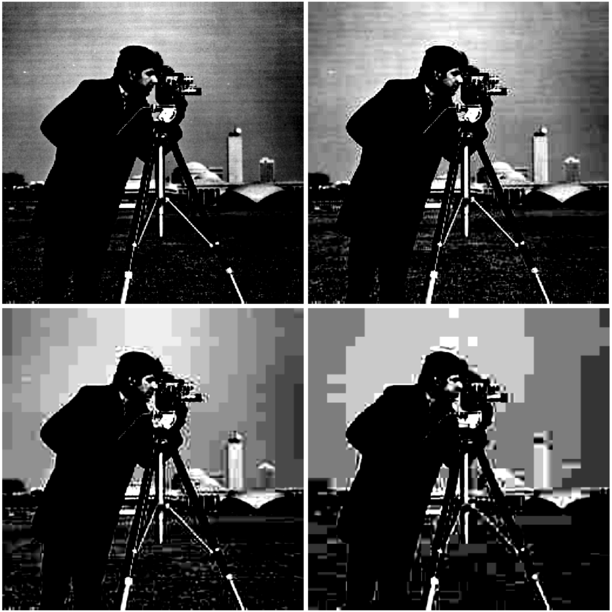
✔ **Reconstruction errors:**

−5	−2	0	1	1	−1	−1	−1
−4	1	1	2	3	0	0	0
−5	−1	3	5	0	−1	0	1
−1	0	1	−2	−1	0	2	4
−4	−3	−3	−1	0	−5	−3	−1
−2	−2	−3	−3	−2	−3	−1	0
2	1	−1	1	0	−4	−2	−1
4	3	0	0	1	−3	−1	0

✔ **Block RMS error = *2.26***

✔ **PSNR = *41 dB***

JPEG - example (higher contrast)

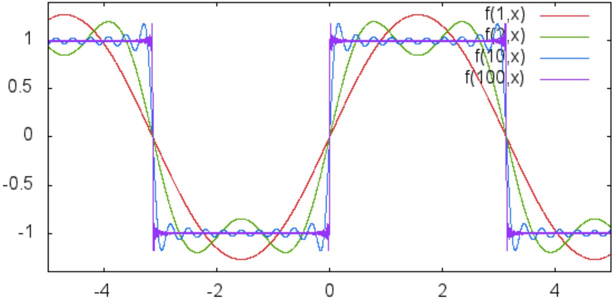


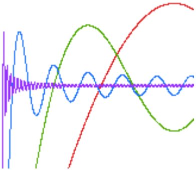
Gibbs phenomenon

The Gibbs phenomenon is the result of the approximation of a signal by its partial sums.

If the function f presents a discontinuity, then its partial representation will present oscillations and overshoots by the jump:

- the amplitude of the overshoots **does not depend on the number of coefficients** in the partial sum,
- and as we increase the number of terms in the sum **the frequency of the oscillations increases**.





Gibbs phenomenon

A periodic function with a jump f can be decomposed as the sum $f = \tilde{f} + s$ where $\tilde{f}$ is continuous and s is a jump function.
By the localization principle, the Fourier series of the continuous part $\tilde{f}$ converges everywhere. And the Gibbs phenomenon will affect only the jump function s.

Therefore, to characterize the Gibbs phenomenon it suffices to study one jump function. We're going to study the 2π-periodic sawtooth function.

Gibbs phenomenon

Proposition 5.2: (Gibbs phenomenon)

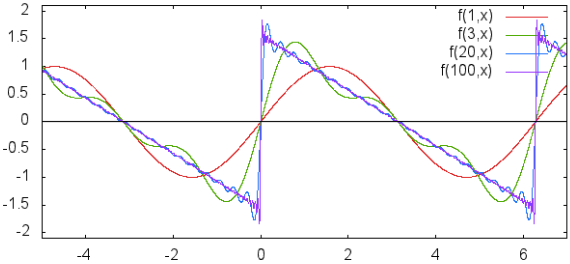
Consider the partial sums of the 2π-periodic function $s(x) = \frac{\pi - x}{2}$

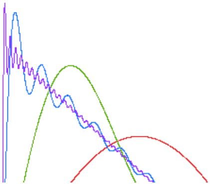
$$s_n(x) := \sum_{k=1}^{n} \frac{\sin(kx)}{k},$$

then

$$\forall \varepsilon > 0 \quad \lim_{n\to\infty} \sup_{x\in(0,\varepsilon]} s_n(x) = (1+c)s(0^+),$$

$$\lim_{n\to\infty} \inf_{x\in[-\varepsilon,0)} s_n(x) = (1+c')s(0^-).$$





Gibbs phenomenon

Démonstration On va étudier la suite $s_n(\frac{\pi}{n})$ quand $n \to \infty$. On commence par étudier les variations de $\boxed{G(a) =: \int_0^a \frac{\sin(t)}{t} dt}$ pour en déduire que $G(\pi) > G(+\infty)$. La fonction $G(a)$ est croissante sur les intervalles pairs $[2k\pi, (2k+1)\pi]$ et décroissante sur les intervalles impairs. On voit aisément que $|G((n+1)\pi) - G(n\pi)|$ est une suite décroissante. Il en résulte que la suite $G(2n\pi)$ est une suite croissante strictement, la suite $G((2n+1)\pi)$ une suite strictement décroissante, et les deux convergent vers une valeur commune notée $G(+\infty)$. On a donc $G(\pi) > G(+\infty)$. On sait par ailleurs que $G(+\infty) = \frac{\pi}{2}$. Revenons à la suite $s_n(\frac{\pi}{n})$. On a

$$s_n(\frac{\pi}{n}) = \sum_{k=1}^{n} \frac{\sin(\frac{k\pi}{n})}{k} = \frac{\pi}{n} \sum_{k=1}^{n} \frac{\sin(\frac{k\pi}{n})}{\frac{k\pi}{n}} \to_{n\to+\infty} \int_0^{\pi} \frac{\sin u}{u} du.$$

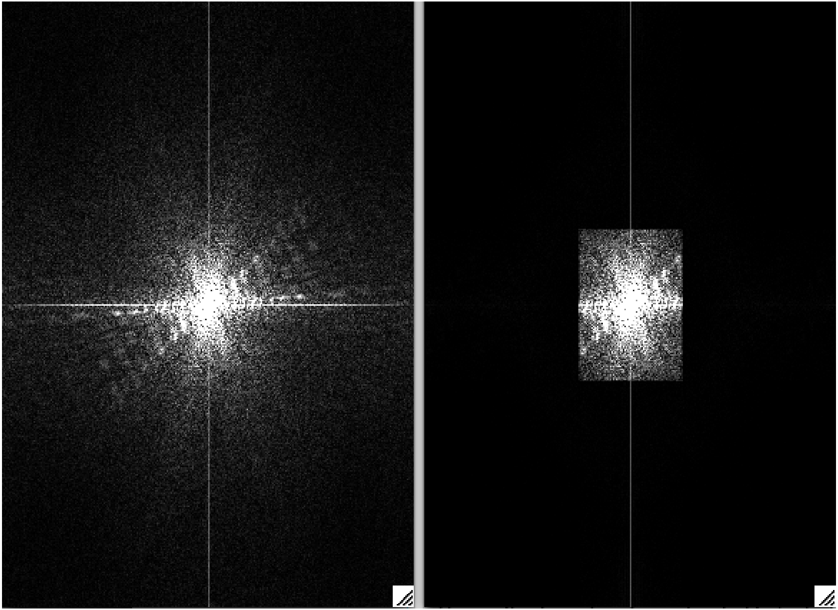
La dernière limite vient du fait que l'on reconnaît la somme de Riemann associée à l'intégrale. Mais

$$s_n(\frac{\pi}{n}) \to G(\pi) > G(+\infty) = \frac{\pi}{2} = s(0^+),$$

car $s(0^+) = \frac{\pi}{2} = \int_0^{+\infty} \frac{\sin u}{u} du$. Donc pour tout n, il y a une valeur très proche de 0, en l'occurrence $\frac{\pi}{n}$, telle que la somme partielle de la série de Fourier dépasse d'un facteur constant $\frac{G(\pi)}{G(+\infty)}$ la valeur de la limite $s(0^+)$. Pour raisons de symétrie, la même chose se produit en 0^- avec la suite $s_n(-\frac{\pi}{n})$. Nous avons donc montré l'existence des limites sup et inf de l'équation (5.3).

Gibbs phenomenon in images

Gibbs phenomenon in images



Summary

- L^1, L^p, L^∞ spaces and norms
  - Completeness (Banach) and density results
- Hilbert spaces
  - Orthogonality and projections
  - Riesz theorem for linear maps
- Hilbert bases
  - Parseval's identity : $f \in H$ then $\|f\|^2 = \sum_n |c_n(f)|^2$
  - Haar base
  - Fourier base of $L^2(-\pi, \pi)$: $\frac{1}{\sqrt{2\pi}} \left(e^{inx}\right)_{n\in\mathbb{Z}}$
  - Weak convergence: $u_n \rightharpoonup u$ if $\forall v \in H, (v, u_n) \to (v, u)$
- Fourier series
  - L^2 convergence of partial sums
- Convolution and Fourier
  - Fundamental theorem of signal processing: all translation invariant linear operators are convolutions
  - Convolution theorem: $c_n(f * g) = c_n(f)c_n(g)$