Hilbert and Fourier analysis
C6

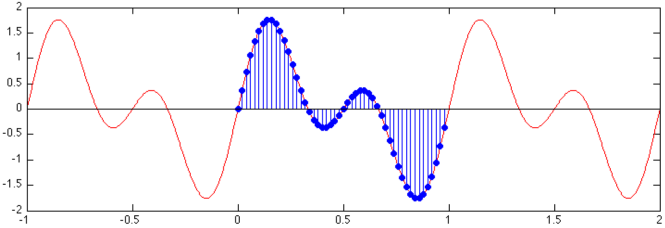
M. Delbracio & G. Facciolo

Today's topics

- Discrete Fourier Transform
  - Forward and inverse DFT
  - Aliasing and subsampling
- Image Manipulation through DFT
  - Fourier image representation
  - Zoom by zero-padding
  - Image Traslation, Rotation

Let u be a periodic, continuous signal with period a for which we know N (odd) samples uniformly distributed:

$$u\left(\frac{ka}{N}\right) := u_k, \quad k = 0, \ldots, N-1$$



Suppose we look for a trigonometric polynomial

$$P(x) = \sum_{n=-\frac{N}{2}}^{n=\frac{N}{2}-1} \tilde{u}_n \mathrm{e}^{\frac{2\pi inx}{a}}$$

that interpolates the samples u_k, that is:

$$P\left(\frac{ka}{N}\right) = u_k \quad k = 0, \ldots, N-1$$

How do we find this polynomial?

Auxiliary Properties of ω_N

Let us call $\omega_N = \mathrm{e}^{\frac{2\pi i}{N}}$, Nth-rooth of unity : $\omega_N^N = 1$.

(a) $\sum_{k=0}^{N-1} \omega_N^k = 0$

(b) $\sum_{k=0}^{N-1} \omega_N^{kl} = 0$ for $l \neq 0$ module N

(c) $\sum_{k=k_0}^{k_0+N-1} \omega_N^{kl} = 0$ for $l \neq 0$ module N.

Excercise: prove it.

Discrete Fourier Transform DFT

Let us call $\omega_N = \mathrm{e}^{\frac{2\pi i}{N}}$, Nth-rooth of unity : $\omega_N^N = 1$.

Def 1: The *Discrete Fourier Transform* DFT of u_k is defined as the sequence

$$\tilde{u}_n = \frac{1}{N} \sum_{k=0}^{N-1} u_k \omega_N^{-k \cdot n} \quad \text{for} \quad n = -\frac{N}{2}, \ldots, \frac{N}{2} - 1$$

and the *Inverse Discrete Fourier Transform* IDFT as

$$u_k = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} \tilde{u}_n \omega_N^{n \cdot k} \quad \text{for} \quad k = 0, \ldots, N-1.$$

Proposition 1: The Discrete Fourier Transform of u_k, namely $\tilde{u}_n$ are the unique coefficients such that the trigonometric polynomial

$$P(x) = \sum_{n=-\frac{N}{2}}^{n=\frac{N}{2}-1} \tilde{u}_n \mathrm{e}^{\frac{2\pi i n x}{a}}$$

satisfies

$$P\left(\frac{ka}{N}\right) = u_k \quad \text{for} \quad k = 0, \ldots, N-1$$

In other words the DFT composed with the IDFT gives the Identity:

$$u_k \xrightarrow{\text{DFT}} \tilde{u}_n \xrightarrow{\text{IDFT}} u_k$$

Proof.

For $k = 0, \dots, N-1$,

$$P\left(\frac{ka}{N}\right) = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} \tilde{u}_n \mathrm{e}^{\frac{2\pi inka}{Na}} = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} \tilde{u}_n \omega_N^{nk}$$

$$= \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} \left(\frac{1}{N}\sum_{l=0}^{N-1} u_l \omega_N^{-ln}\right)\omega_N^{nk}$$

$$= \frac{1}{N}\sum_{l=0}^{N-1} u_l \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} \omega_N^{-ln}\omega_N^{nk}$$

$$= \frac{1}{N}\sum_{l=0}^{N-1} N\delta(k-l)u_l = u_k$$

Where δ is defined over $\mathbb{Z}$, and takes value 1 in 0 and 0 otherwise.

The unicity comes from the fact that every surjective linear application in $\mathbb{C}^N$ to itself is also injective.

$\square$

Corollary 1: If u is a trigonometric polynomial

$$u(x) = \sum_{n=-\frac{N}{2}}^{n=\frac{N}{2}-1} \tilde{u}_n \mathrm{e}^{\frac{2\pi i n x}{a}}$$

then $\tilde{u}_n$ can be calculated from the DFT of the samples of u.

In this particular case the DFT of u_k are the Fourier coefficients of u: $\tilde{u}_n = c_n(u)$.

In other words after sampling a function u, the DFT computes the Fourier coefficients of u from the samples u_k as long as u is a trigonometric polynomial of degree $\leq \frac{N}{2}$.

Proposition 2: Let u be a continuous a-periodic function. Then the coefficients $\tilde{u}_n$ approximate the Fourier coefficients $c_n(u)$ by the trapezoid method, for $n = -\frac{N}{2}, \ldots, \frac{N}{2} - 1$.

Remember that if $u \in L^2(0, a)$, the coefficients from the Fourier series of u are defined for $n \in Z$, by

$$c_n(u) = \frac{1}{a} \int_0^a u(x) \mathrm{e}^{-\frac{2i\pi nx}{a}}$$

$$\tilde{u}_n = \frac{1}{N} \sum_{k=0}^{N-1} u_k \omega_N^{-nk}$$

Proof.
Excercise (approximate integral by trapezoids + u is a-periodic) $\square$

Proposition 3: If the samples u_k are real then:

- $\tilde{u}_0$ and $\tilde{u}_{-\frac{N}{2}}$ are real
- $\tilde{u}_k = \overline{\tilde{u}_{-k}}$ for $k = 1, \dots, \frac{N}{2} - 1$.

Proof.

By definition

$$\tilde{u}_n = \frac{1}{N} \sum_{k=0}^{N-1} u_k \omega_N^{-nk}$$

We have $\tilde{u}_0 = \frac{1}{N} \sum_k u_k$ and $\tilde{u}_{-\frac{N}{2}} = \frac{1}{N} \sum_k (-1)^k u_k$ which are both real.

Moreover,

$$\tilde{u}_{-n} = \frac{1}{N} \sum_{k=0}^{N-1} u_k \omega_N^{kn} = \frac{1}{N} \sum_{k=0}^{N-1} u_k \overline{\omega_N^{-nk}} = \overline{\frac{1}{N} \sum_{k=0}^{N-1} u_k \omega_N^{-nk}} = \overline{\tilde{u}_n}$$

$\square$

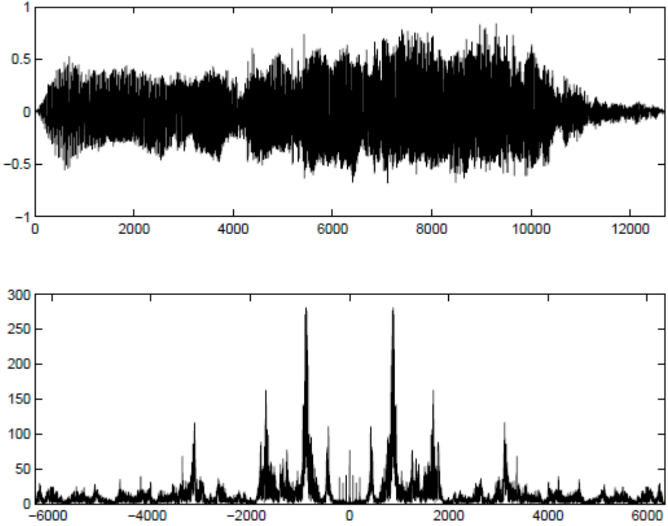
Proposition 4: If u is a real trigonometric polynomial with frequencies between: $-\frac{N}{2}, \ldots, \frac{N}{2}-1$ the coefficient $\tilde{u}_{-\frac{N}{2}}$ is zero.

Proof.

$$P(x) = \tilde{u}_0 + \sum_{n=1}^{\frac{N}{2}-1} \left(\tilde{u}_n \mathrm{e}^{\frac{2in\pi x}{a}} + \tilde{u}_{-n} \mathrm{e}^{-\frac{2in\pi x}{a}} \right) + \tilde{u}_{-\frac{N}{2}} \mathrm{e}^{-\frac{2iN\pi x}{a}}$$

where every term is real with the exception of the last one, so we necessary have $\tilde{u}_{-\frac{N}{2}} = 0$. $\square$

Audio signal: "ah" vowel



The Two-Dimensional DFT

Consider $a \in \mathbb{R}$ and u continuous function $u : \mathbb{R}^2 \to \mathbb{R}$ such that $u(x + a, y + a) = u(x, y)$. We choose $N \in \mathbb{N}$ (odd) and we define

$$u_{k,l} = u\left(\frac{ka}{N}, \frac{la}{N}\right).$$

The 2D DFT of $u_{k,l}$ is defined as the sequence of coefficients, for $m, n \in \{-\frac{N}{2}, \dots, \frac{N}{2} - 1\}$,

$$\tilde{u}_{m,n} = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} u_{k,l} \omega_N^{-mk} \omega_N^{-nl}.$$

Proposition 1:

The Discrete Fourier Transform of $u_{k,l}$, namely $\tilde{u}_{m,n}$ are the unique coefficients such that the trigonometric polynomial

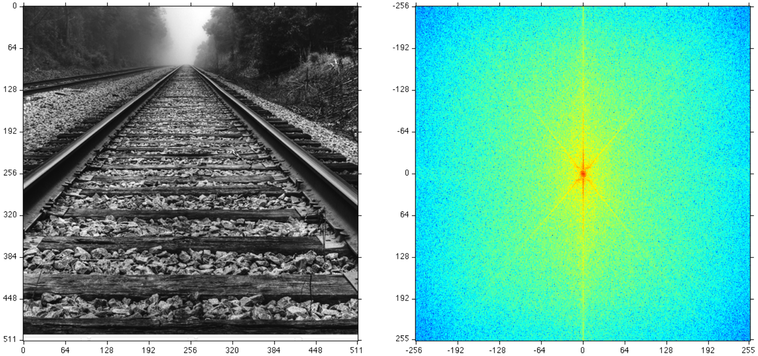
$$P(x, y) = \sum_{n=-\frac{N}{2}}^{m,n=\frac{N}{2}-1} \tilde{u}_{m,n} \mathrm{e}^{\frac{2\pi imx}{a}} \mathrm{e}^{\frac{2\pi iny}{a}}$$

satisfies

$$P\left(\frac{ka}{N}, \frac{la}{N}\right) = u_{k,l} \quad \text{for} \quad k, l = 0, \ldots, N-1.$$

In other words:

- the 2D DFT composed with the 2D IDFT gives the Identity.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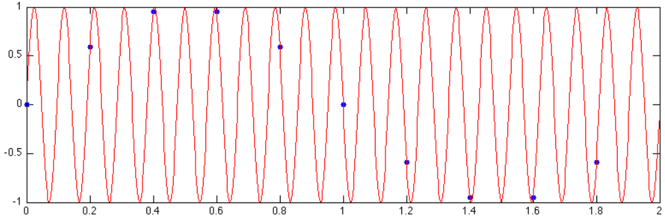
Aliasing: repliement de spectre ou aliasage

Aliasing from Wikipedia

"In signal processing and related disciplines, aliasing refers to an effect that causes different signals to become indistinguishable (or aliases of one another) when sampled. It also refers to the distortion or artifact that results when the signal reconstructed from samples is different from the original continuous signal."



Aliasing Theorem

Theorem 1: Let $u \in L^2(0, a)$ such that $\sum_{n\in\mathbb{Z}} |c_n(u)| < \infty$. Then the DFT of the N samples (u_k) is the N-periodization of the Fourier coefficients of u,

$$\tilde{u}_n = \sum_{q=-\infty}^{+\infty} c_{n+qN}(u), \quad \text{for} \quad n = -\frac{N}{2}, \dots, \frac{N}{2} - 1.$$

Proof.

Since $u(x) = \sum_{m\in\mathbb{Z}} c_m(u)\mathrm{e}^{\frac{2\pi i x m}{a}}$ we have

$$u\left(\frac{ka}{N}\right) = \sum_{m\in\mathbb{Z}} c_m(u)\mathrm{e}^{\frac{2\pi i m}{a}\frac{ka}{N}} = \sum_{m\in\mathbb{Z}} c_m(u)\omega_N^{mk}, \quad \text{for} \quad k = 0, \dots, N-1$$

Every $m \in \mathbb{Z}$ can be written as $m = qN + n$ with $-\frac{N}{2} \leq n \leq \frac{N}{2} - 1$. Hence for $k = 0, \dots, N-1$,

$$u\left(\frac{ka}{N}\right) = \sum_{q=-\infty}^{+\infty} \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} c_{n+qN}(u)\omega_N^{k(n+qN)} = \sum_{q=-\infty}^{+\infty} \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} c_{n+qN}(u)\omega_N^{kn}\underbrace{\omega_N^{Nq}}_{=1}$$

$$= \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} \left(\sum_{q=-\infty}^{+\infty} c_{n+qN}(u)\right)\omega_N^{kn}$$

But from the inverse DFT formula, $u\left(\frac{ka}{N}\right) = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} \tilde{u}_n\omega_N^{nk}$ since both equations define the DFT we have the desired result

$$\tilde{u}_n = \sum_{q=-\infty}^{+\infty} c_{n+qN}(u), \text{ for } n = -\frac{N}{2}, \dots, \frac{N}{2} - 1.$$

□

Aliasing Theorem : analogous result in 2D

Theorem 2: Let $u \in L^2([0,a]^2)$ such that $\sum_{n,m\in\mathbb{Z}} |c_{n,m}(u)| < \infty$. Then the DFT of the N samples $(u_{k,l})$ is the (N,N)-periodization of the Fourier coefficients of u,

$$\tilde{u}_{m,n} = \sum_{p,q=-\infty}^{+\infty} c_{m+pN,n+qN}(u), \quad \text{for} \quad m,n = -\frac{N}{2},\ldots,\frac{N}{2}-1.$$

Aliasing

The term aliasing comes from the presence of parasite coefficients

$$c_{m+pN,n+qN}(u) \quad \text{for} \quad (p,q) \neq (0,0)$$

when computing the frequency coefficient $\tilde{u}_{m,n}$.

$$\tilde{u}_{m,n} = c_{m,n}(u) + \underbrace{\sum_{p,q \neq (0,0)} c_{m+pN,n+qN}(u)}_{\text{alias}}$$

for $m, n = -\frac{N}{2}, \ldots, \frac{N}{2} - 1$,

To think: When does aliasing occur?

Subsampling

Given a sampled signal we subsample it a factor p.

Def 2: Let (u_k), $k = 0, \ldots, N-1$ and $p \in \mathbb{N}$ dividing N. We define the *subsampling operator of order p* as follows:

$$S_p : \mathbb{R}^N \to \mathbb{R}^{N/p}$$

$$(u_k)_{k=0,\ldots,N-1} \longrightarrow (v_k) = (u_{kp})_{k=0,\ldots,N/p-1}$$

The signal (v_k) is called a subsampling of order p.

Subsampling

A *typical* case $p = 2$. N is even and $N/2$ is even.

Corollary 2: Let $(v_k) = S_2((u_k))$. Then $(\tilde{v}_n)$, the DFT of (v_k) can be written, for $n = -\frac{N}{4}, \ldots, \frac{N}{4} - 1$,

$$\tilde{v}_n = \tilde{u}_n + \underbrace{\tilde{u}_{n-\frac{N}{2}}}_{=0 \text{ if } n<0} + \underbrace{\tilde{u}_{n+\frac{N}{2}}}_{=0 \text{ if } n\geq 0}.$$

Proof.

Let $P(x)$ be the *unique* trigonometric polynomial with N coefficients that interpolates $(u_k)_{k=0,\ldots,N-1}$. Since P is a trigonometric polynomial with N coefficients,

$$\tilde{u}_n = c_n(P) \quad \text{for} \quad n = -\frac{N}{2}, \ldots, \frac{N}{2} - 1$$

Then apply *Theorem 1* to $P(x)$. The DFT of the $\frac{N}{2}$ samples (v_k) satisfies

$$\tilde{v}_n = \sum_{q=-\infty}^{+\infty} c_{n+q\frac{N}{2}} = \begin{cases} c_n(P) + c_{n+\frac{N}{2}}(P) & \text{if } n < 0, \\ c_n(P) + c_{n-\frac{N}{2}}(P) & \text{if } n \geq 0. \end{cases}$$

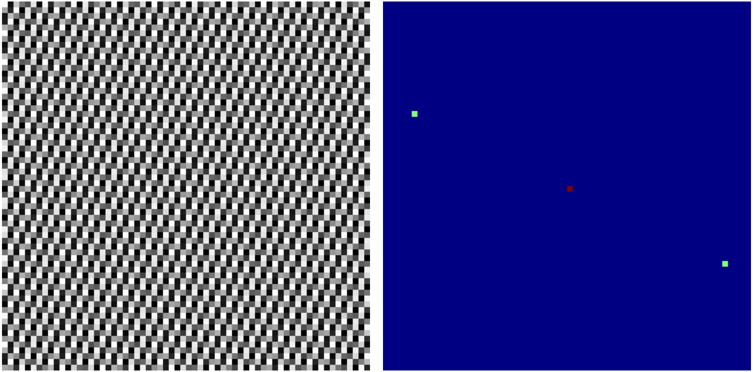
Subsampling - Aliasing: 2D

Corollary 3: Let $(v_{k,l}) = S_2\left((u_{k,l})\right)$ a subsampled digital image by a factor of 2. Then $(\tilde{v}_{m,n})$, the DFT of $(v_{k,l})$ can be written, for $m, n = -\frac{N}{4}, \ldots, \frac{N}{4} - 1$ as

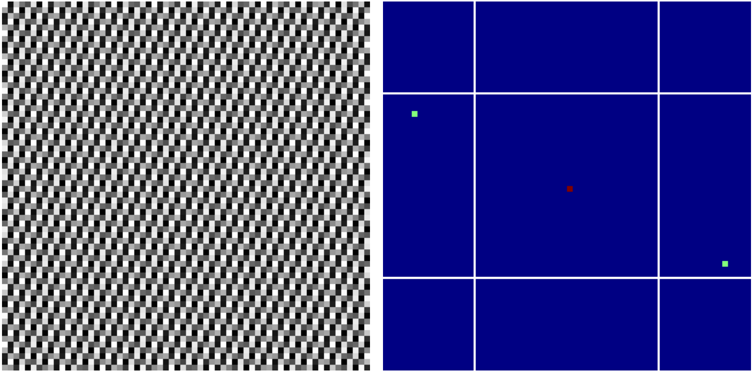
$$\tilde{v}_{m,n} = \sum_{(\epsilon_1,\epsilon_2)\in\{0,1,-1\}} \tilde{u}_{m+\epsilon_1\frac{N}{2},n+\epsilon_2\frac{N}{2}}.$$

What does the previous Corollary say?

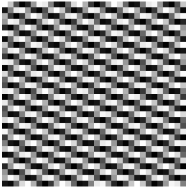
Aliasing: Toy Example $2\times$ subsampling

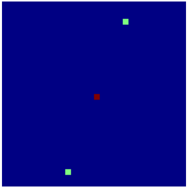


Aliasing: Toy Example 2× subsampling

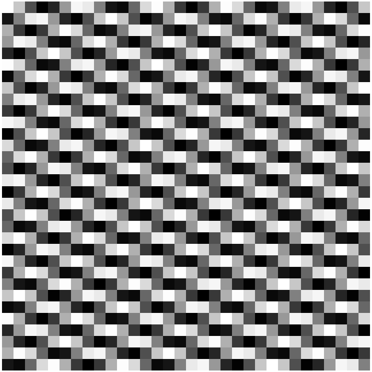


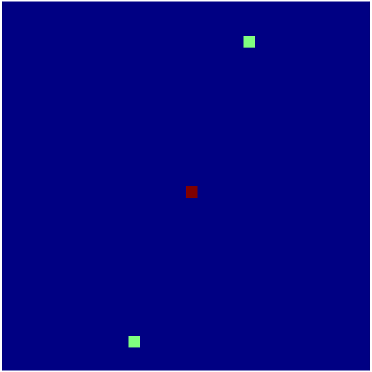
Aliasing: Toy Example $2\times$ subsampling



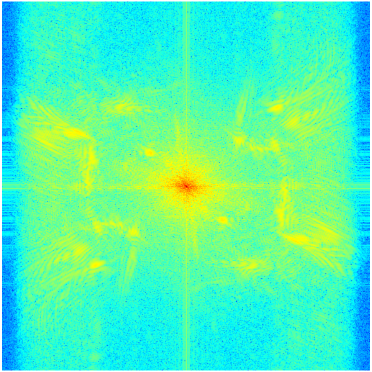


Aliasing: Toy Example 2× subsampling

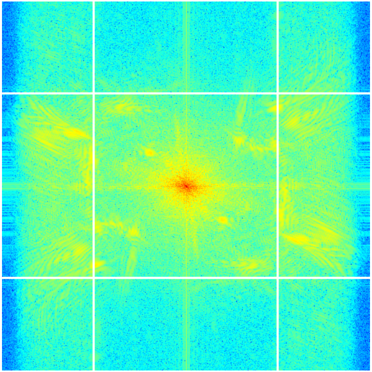




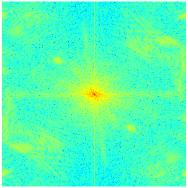
Aliasing: Barbara 2× subsampling



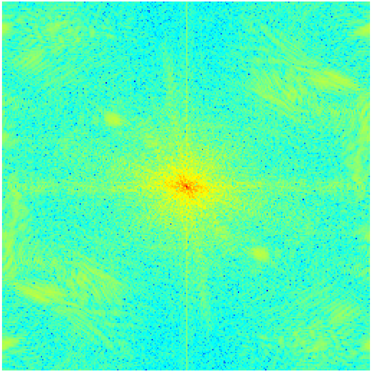
Aliasing: Barbara $2\times$ subsampling



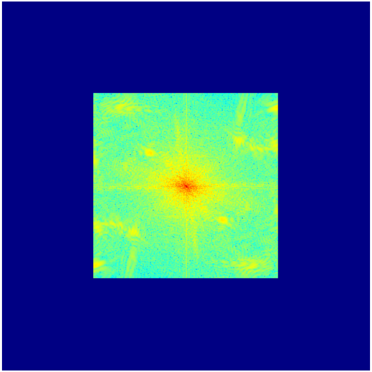
Aliasing: Barbara $2\times$ subsampling



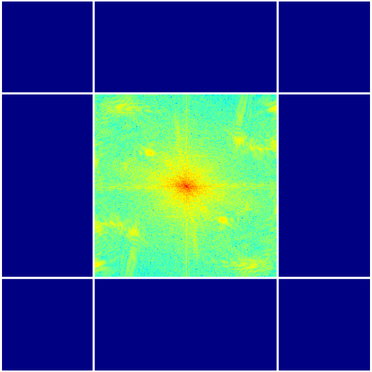
Aliasing: Barbara $2\times$ subsampling



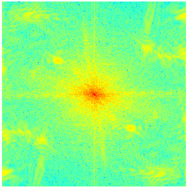
Lowpass filter before: Barbara 2$\times$ subsampling



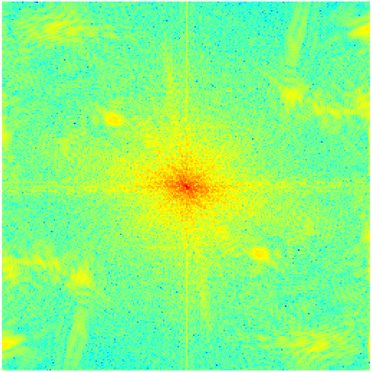
Lowpass filter before: Barbara 2× subsampling



Lowpass filter before: Barbara $2\times$ subsampling

Lowpass filter before: Barbara $2\times$ subsampling



Fast Fourier Transform

How many operations do we need to compute the DFT of a sequence of size N?

$$\tilde{u}_n = \sum_{k=0}^{N-1} u_k \omega_N^{-nk}$$

N^2 operations. (one addition + one multiplication = one operation).

However, the Fast Fourier Transform FFT can do it in :

$$\approx N \log N \quad \text{operations}$$

$\boxed{\text{TD Excercise!}}$

Image Representation from DFT

Let $(\tilde{u}_{m,n})_{m,n=-\frac{N}{2},\ldots,\frac{N}{2}-1}$ be a set of Fourier coefficients.
We'd like to represent an image that has this Fourier coefficients.
For that, we fix the image domain, e.g. $[0,a]^2$ and we set

$$P(x,y) = \sum_{m,n=-\frac{N}{2}}^{\frac{N}{2}-1} \tilde{u}_{m,n} \mathrm{e}^{\frac{2\pi i m x}{a}} \mathrm{e}^{\frac{2\pi i n y}{a}}.$$

This image is a-periodic and its samples are

$$u_{k,l} = u\left(\frac{ka}{N}, \frac{la}{N}\right) = \sum_{m,n=-\frac{N}{2}}^{\frac{N}{2}-1} \tilde{u}_{m,n} \omega_N^{mk} \omega_N^{nl}$$

Also the DFT of $(u_{k,l})$ are the coefficients $(\tilde{u}_{m,n})$.

Traslation by DFT

Recall that if we know the Fourier coefficients of a signal $\tilde{u}_n$ we can compute the associated trigonometric polynomial

$$P(x) = \sum_{n=\frac{-N}{2}}^{\frac{N}{2}-1} \tilde{u}_n \mathrm{e}^{\frac{2\pi inx}{a}}$$

then

$$\tau_\alpha P(x) = P(x-\alpha) = \sum_{n=\frac{-N}{2}}^{\frac{N}{2}-1} \tilde{u}_n \mathrm{e}^{\frac{2\pi inx}{a}} \mathrm{e}^{-\frac{2\pi in\alpha}{a}}$$

The DFT of $P(x-\alpha)$, $\tilde{v}_n$, is calculated from the DFT of $P(x)$,$\tilde{u}_n$, by:

$$\tilde{v}_n = \tilde{u}_n \mathrm{e}^{-\frac{2\pi in\alpha}{a}}$$

Rotation by DFT

Method Yarovslasky: three pass

To calculate a rotation of angle θ of image $u(i,j)$, we have to compute $u(R_{-\theta}(i,j))$,

$$R_{-\theta} := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & \tan\frac{\theta}{2} \\ 0 & 1 \end{pmatrix}}_{:=T(\theta)} \underbrace{\begin{pmatrix} 1 & 0 \\ -\sin\theta & 1 \end{pmatrix}}_{:=S(\theta)} \begin{pmatrix} 1 & \tan\frac{\theta}{2} \\ 0 & 1 \end{pmatrix}$$

$$R_{-\theta} := T_\theta S_\theta T_\theta$$

Notice that T_θ (S_θ) is a *line by line translation* (column by column):

$$(T_\theta u)(i,j) = u\left(i + j\tan\frac{\theta}{2}, j\right)$$

- By doing 1D translations we can do an image rotation: three-pass algorithm.
- We can use the DFT for the 1D translations: $\boxed{\text{TP sessions!}}$

Rotation by DFT

Method Yarovslasky: three pass

click

click

Bilinear

Three-pass

Zoom by DFT

"Zero-padding interpolation" (prolongement par des zéros)

Let (u_k) be a digital sequence of size N and suppose we want to interpolate it a factor of 2 (generate a sequence of size $2N$).

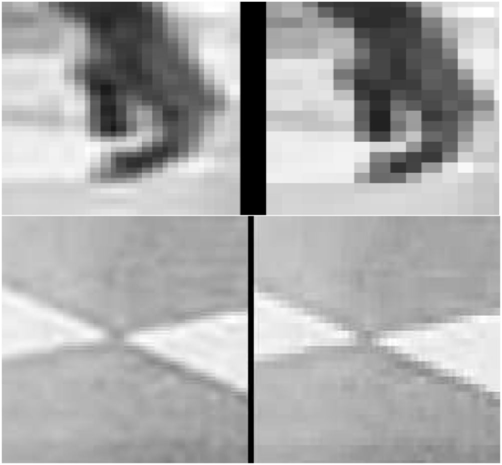
We define $(v_k)_{k=0,\dots 2N-1}$ the signal whose DFT $\tilde{v}_n$ is:

$$\begin{aligned} \tilde{v}_n = \tilde{u}_n \quad & \text{if} \quad -\frac{N}{2} \le n \le \frac{N}{2} - 1 \\ = 0 \quad & \text{if} \quad n \in [-N, -\frac{N}{2} - 1] \cup [\frac{N}{2}, N-1] \end{aligned}$$

Then (v_k) verifies: $v_{2k} = u_k$ for $k = 0, \dots, N-1$.

$$v_{2k} = \sum_{n=-N}^{N-1} \tilde{v}_n \omega_{2N}^{2nk} = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} \tilde{u}_n \omega_N^{nk} = u_k$$

Zoom by DFT



left: DFT zero-padding - right: nearest-neigbhour interpolation

We haven't paid any attention to the phase information of the DFT.

We will discuss it on the TP1.

A

B

mod(DFT(A)) and phase(DFT(B))

mod(DFT(B)) and phase(DFT(A))