Hilbert and Fourier analysis
C7

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Today's topics

- Distributions Theory
  - Motivation
  - Definition, Convergence
  - Examples
  - Why are they called *Generalized Functions*?
  - Derivative, differentiation formula
  - Fourier Series, Poisson Formula

Motivation...

Assume we are given a function $u \in C^k(\mathbb{R})$ and $\varphi \in C_c^\infty$ an auxiliary function that we will call "test function". Let $u^{(k)}$ be the kth-derivative

of u, then from the integration by parts

$$\int u^{(k)}(x)\varphi(x)\,\mathrm{d}x = (-1)^k \int u(x)\varphi(x)^{(k)}\,\mathrm{d}x$$

There are no boundary terms, since every "test function" φ has compact support in $\mathbb{R}$.

Now, If $u \notin C^k$ but $u \in L^1_{\text{loc}}(\mathbb{R})$ the right side still makes sense.

$\boxed{\text{How can we use this to generalize the } \textit{derivative} \text{ for } u \in L^1_{\text{loc}}(\mathbb{R})\text{?}}$

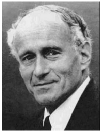
We can define $u^{(k)}$ by means of the application to every *test function*

$$l_{u^{(k)}} : \varphi \longrightarrow (-1)^k \int u(x)\varphi(x)^{(k)}\,\mathrm{d}x$$

$l_{u^{(k)}}$ is not a function but is a linear form (we will call them *generalized functions* or *distributions*).

$\boxed{\text{We can define } \textit{something} \text{ by the application to a set of } \textit{test functions}\text{!}}$

Laurent Schwartz



En 1944, il a, une nuit, une illumination : depuis longtemps les mathématiciens cherchaient à légitimer les calculs faits par les physiciens comme Dirac ou Heaviside, et qui utilisent des fonctions très étranges, par exemple une fonction valant 0 partout, sauf en un point où elle vaut plus l'infini, et d'intégrale 1. Cette nuit-là, Schwartz invente une notion de fonction généralisée, les distributions. Il développera ensuite pendant 4 ans cette théorie, qui est à la fois simple, élégante, et très puissante: les distributions ont joué un rôle crucial dans le développement des équations aux dérivées partielles, mais furent aussi employées en analyse de Fourier ou en théorie du potentiel. C'est aussi une des rares théories mathématiques du XXème siècle qui puisse être enseignée à l'université à des niveaux raisonnables. Pour cette théorie, Schwartz recevra en 1950 la prestigieuse médaille Fields (il est alors le premier Français à recevoir cette récompense). D'ailleurs, Schwartz aura beaucoup de difficultés pour se rendre aux Etats-Unis pour recevoir cette médaille en raison de son passé trotskiste.

Laurent Schwartz (5 mars 1915 à Paris - 4 juillet 2002)
Théorie des Distributions (1950)

Def 1: We call *distribution* on Ω (open set in $\mathbb{R}^n$) to every linear form u over $\mathcal{C}_c^\infty(\Omega)$ satisfying the following continuous property: for every compact set $K \subset \Omega$ there exists an integer $p = p(K)$ and a constant $C = C(K)$ such that for every function $\varphi \in \mathcal{C}_c^\infty(\Omega)$ with support included in K, we have

$$| < u, \varphi > | \leq C \sup_{|i| \leq p, x \in K} |\partial^i \varphi(x)|.$$

- We call $\mathcal{D}(\Omega) := C_c^\infty(\Omega)$ (test functions)
- We will refer to the vectorial space of Distributions as $\mathcal{D}'(\Omega)$ the *dual* of $C_c^\infty(\Omega)$.
- If **p** can be chosen independently of **K**, we say that the distribution is of finite order. The smallest value **p** is called the order of **u**.
- u is continuous: Given a sequence of test functions (φ_j) with same support K that converges uniformly and also all their derivatives, then

$$< u, \varphi_j > \longrightarrow < u, \varphi > .$$

Distributions: Examples

If $u \in L^1_{\text{loc}}(\Omega)$ then

$$< \tilde{u}, \varphi > := \int_\Omega u(x)\varphi(x)\,\mathrm{d}x$$

Is this a Distribution? It is a linear form

and for every $K \subset \Omega$ compact and for $\varphi \in \mathcal{C}^\infty_K$, we have

$$| < \tilde{u}, \varphi > | \leq \int_K |u(x)||\varphi(x)|\mathrm{d}x \leq \left(\int_K |u(x)|\,\mathrm{d}x \right) \left(\sup_{x \in K} |\varphi(x)| \right).$$

Since,

$$| < \tilde{u}, \varphi > | \leq \underbrace{\left(\int_K |u(x)|\,\mathrm{d}x \right)}_{=C} \left(\sup_{x \in K} |\varphi(x)| \right).$$

it is a distribution of order 0 (p independent of K, $p = 0$).

Excercise
Using Corolary 3.1 show that if two distributions $\tilde{u}$ and $\tilde{v}$ in $\mathbb{R}^n$ associated to functions u and v in L^1_{loc}, are equal ($\tilde{u} = \tilde{v}$) then $u = v$ *a.e.*

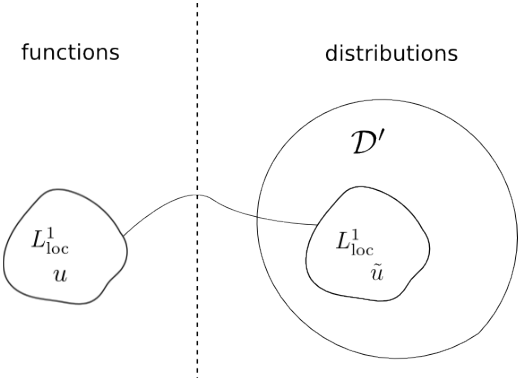
Thanks to this unicity result,

$$\tilde{u} = \tilde{v} \text{ in } \mathcal{D}'(\Omega) \Longleftrightarrow u = v \text{ in } L^1_{\text{loc}}(\Omega)$$

we can consider the functions $u \in L^1_{\text{loc}}$ as the distributions (u and $\tilde{u}$).

$L^1, L^2, L^\infty \subset L^1_{\text{loc}}$

$L^\infty_{\text{loc}} \subset L^2_{\text{loc}} \subset L^1_{\text{loc}}$



Dirac delta *function*

Dirac distribution in $x \in \mathbb{R}^n$ is defined by

$$< \delta_x, \varphi >= \varphi(x), \quad \forall \varphi \in C_c^\infty(\Omega)$$

- it is a linear form
- recall the definition of distribution, it's valid for $p = 0$. (zero order)

In the same spirit we can define the *Dirac comb* (peigne de Dirac):

$$u = \sum_{\mathbf{n} \in \mathbb{Z}^n} \delta_{2\mathbf{n}\pi}$$

Remark: the application of u to a test function gives the addition of a finite number of values due to the finite support of φ:

$$< \sum_{\mathbf{n} \in \mathbb{Z}^n} \delta_{2\mathbf{n}\pi}, \varphi >= \sum_{\mathbf{n} \in \mathbb{Z}^n} \varphi(2\mathbf{n}\pi)$$

Also, we can define (**Excercise**)

$$< \partial^{\mathbf{i}} \delta_x, \varphi >:= (-1)^{|\mathbf{i}|} \partial^{\mathbf{i}} \varphi(x)$$

Convergence of Distributions

Def 2: Let (u_n) be a sequence of distributions on Ω and u one distribution on Ω. We say that u_n converges to u in the *distribution sense* if for every $\varphi \in \mathcal{C}_c^\infty(\Omega)$ the numerical sequence $(< u_n, \varphi >)_n$ converges to $< u, \varphi >$. We write

$$u_n \xrightarrow{\mathcal{D}'(\Omega)} u$$

Theorem 1: If (u_n) is a sequence of distributions on Ω such that $\forall \varphi \in \mathcal{C}_c^\infty$ the sequence $< u_n, \varphi >$ converges to a number, that we note $< u, \varphi >$ then, the limit u, is also a distribution in $\mathcal{D}'(\Omega)$.

We are going to accept it without proof: it's a consequence of the *Banach Steinhaus Theorem*.

Let E be a Banach Space and $u_j : E \to \mathbb{R}$ linear applications such that $\forall x \in E \quad \sup_j |u_j(x)| < \infty$. Then $\exists K > 0$ such that $\sup_{j,x} |u_j(x)| \leq K$.

Revisiting: Dirac Distribution

Let $u_j \in L^1_{\text{loc}}$ be a sequence of *non-negative* functions, such that $\int u_j = 1$, with $\text{supp}(u_j) \subset B(0, \alpha_j) \subset K$ compact, with $\alpha_j \to 0$. Then

$$u_j \to \delta_0 \quad \text{in the Distribution sense.}$$

Proof.

$$< u_j, \varphi >= \int u_j(x)\varphi(x) = \underbrace{\int u_j(x)\varphi(0)}_{\varphi(0)} - \int u_j(x)(\varphi(0) - \varphi(x))$$

and

$$\begin{aligned}
\left| \int u_j(x)(\varphi(0) - \varphi(x)) \right| &\leq \int u_j(x) |(\varphi(0) - \varphi(x))| \\
&\leq \int u_j(x) \alpha_j \sup |\nabla \varphi| \\
&= \alpha_j \sup |\nabla \varphi|.
\end{aligned}$$

Then

$$< u_j, \varphi > \to \varphi(0) \quad \forall \varphi \in \mathcal{C}_c^\infty.$$

$\square$

Excercise If $u_n \in L^1_{\text{loc}}$ converges to $u \in L^1_{\text{loc}}$, then u_n also converges to u in the distribution sense.

Proof

We have to prove that $\forall \varphi \in C_0^\infty(\Omega)$, $< u_n, \varphi > \to < u, \varphi >$.
Let $\varphi \in C_0^\infty(\Omega)$ with support $K \subset \Omega$. Since

$$< u_n, \varphi > = \int_K u_n(x)\varphi(x)\,\mathrm{d}x$$
$$< u, \varphi > = \int_K u(x)\varphi(x)\,\mathrm{d}x$$

we have that

$$\begin{aligned} | < u_n, \varphi > - < u, \varphi > | &\leq \int_K |u_n(x) - u(x)||\varphi(x)|\,\mathrm{d}x \\ &\leq \sup_{x \in K} |\varphi(x)| \int_K |u_n - u| \stackrel{n\to\infty}{\longrightarrow} 0 \end{aligned}$$

Recall: For sequences in L^2_{loc} (or L^∞_{loc}) the convergence in L^2 (or L^∞) for every compact, implies the convergence in L^1 for every compact and therefore the convergence in the distribution sense.

Derivative of Distributions

Assume we are given a function $u \in C^k(\mathbb{R})$ and $\varphi \in C_c^\infty$ an auxiliary function that we call "test function". Let $u^{(k)}$ be the kth-derivative of u,

then from the integration by parts

$$\int u^{(k)}(x)\varphi(x)\,\mathrm{d}x = (-1)^k \int u(x)\varphi(x)^{(k)}\,\mathrm{d}x$$

There are no boundary terms, since every "test function" φ has compact support in $\mathbb{R}$.

Now, If $u \notin C^k$ but $u \in L^1_{\mathrm{loc}}(\mathbb{R})$ the right side still makes sense.

How can we use this to generalize the *derivative* for $u \in L^1_{\mathrm{loc}}(\mathbb{R})$ or even more for $u \in \mathcal{D}'$?

Def 3: Let $u \in \mathcal{D}'(\Omega)$, we call *partial derivative* of order $\mathbf{i}$ of u in Ω to the distribution $\partial^{\mathbf{i}} u \in \mathcal{D}'(\Omega)$ defined by

$$< \partial^{\mathbf{i}} u, \varphi > := (-1)^{|\mathbf{i}|} < u, \partial^{\mathbf{i}} \varphi > .$$

Derivative of Distributions

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$$< \partial^{\mathbf{i}} u, \varphi > := (-1)^{|\mathbf{i}|} < u, \partial^{\mathbf{i}} \varphi > .$$

We need to check that the definition makes sense (the result is a distribution). It is linear by definition and if u is a distribution

$$\begin{aligned} | < \partial^{\mathbf{i}} u, \varphi > | = | < u, \partial^{\mathbf{i}} \varphi > | &\leq C_K \sup_{x \in K, |\mathbf{j}| \leq p} |\partial^{\mathbf{j}} (\partial^{\mathbf{i}} \varphi(x))| \\ &\leq C_K \sup_{x \in K, |\mathbf{i}+\mathbf{j}| \leq |\mathbf{i}| + p} |\partial^{\mathbf{j}+\mathbf{i}} \varphi(x))| \end{aligned}$$

Remark: We can always differentiate a distribution $u \in \mathcal{D}'(\Omega)$.

Derivative of Distributions

Proposition 1: If $u \in C^k(\Omega)$, then $\partial^{\mathbf{i}}\{u\}$ the derivative in the classical sense coincide to the derivative in the distributions sense $\partial^{\mathbf{i}} u$.

Proof.

If $u \in C^k(\Omega)$ then $\partial^{\mathbf{j}} u$, with $|\mathbf{j}| \leq k$ are in L^1_{loc} so they are distributions. We have,

$$\begin{aligned}
< \partial^{\mathbf{j}}\{u\}, \varphi > &= \int \partial^{\mathbf{j}}\{u\}(x)\varphi(x)\,\mathrm{d}x \\
&= (-1)^{|\mathbf{j}|} \int u(x)\partial^{\mathbf{j}}\varphi(x)\,\mathrm{d}x \\
&= < \partial^{\mathbf{j}} u, \varphi >
\end{aligned}$$

$\square$

Derivative of Distributions

Observations

- We can always differentiate a distribution $u \in \mathcal{D}'(\Omega)$.
- If $u \in C^k(\Omega)$ then $\partial^{\mathbf{j}} u$ with $|\mathbf{j}| \leq k$ is a function and a Distribution!
- If $u \notin C^k(\Omega)$ then $\partial^{\mathbf{j}} u$ is not a function but it is a Distribution!
- We need to learn how to calculate derivatives in the Distribution sense when functions are not differentiable.
- **Remember:** Distributions are called *Generalized Functions*.

Theorem 2: (*Derivative of a limit distribution*)
Let $u_j \in \mathcal{D}'(\Omega)$ be a sequence of distributions such that $u_j \xrightarrow{\mathcal{D}'(\Omega)} u$, then the sequence $\partial^{\mathbf{i}} u_j \xrightarrow{\mathcal{D}'(\Omega)} \partial^{\mathbf{i}} u$

Proof.

$$< \partial^{\mathbf{i}} u_j, \varphi >= (-1)^{|\mathbf{i}|} < u_j, \partial^{\mathbf{i}} \varphi >$$

The right side for $j \to \infty$ converges to:

$$(-1)^{|\mathbf{i}|} < u_j, \partial^{\mathbf{i}} \varphi > \xrightarrow{\mathbb{R}} (-1)^{|\mathbf{i}|} < u, \partial^{\mathbf{i}} \varphi > \overset{def}{=} < \partial^{\mathbf{i}} u, \varphi >$$

$\square$

Remark: Differentiation is always possible and moreover it is a continuous operation! In general this is false in the case of functions.

Differentiation: an example

Let us consider H the Heaviside function:

$$H(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

$$< H', \varphi >= -\int H(x)\varphi'(x)\,\mathrm{d}x = -\int_0^\infty \varphi'(x)\,\mathrm{d}x = \varphi(0)$$

Then,

$$< H', \varphi >= \varphi(0) =< \delta_0, \varphi > .$$

Is this what we would have expected?

Yes! but this can only be explained in the limit by the *Distributions theory*.

Principal Value

Cauchy Principal Value

A function which is not locally integrable is not a distribution. However, in some particular cases, we can define a distribution from a non-locally integrable function.

For example consider,

$$f_\epsilon(x) = \begin{cases} \frac{1}{x} & \text{if } |x| > \epsilon, \\ 0 & \text{if } |x| \leq \epsilon. \end{cases}$$

Then, $f_\epsilon \in L^1_{\text{loc}} \subset \mathcal{D}'(\Omega)$.

We are going to show that f_ϵ converges, in the distribution sense, when $\epsilon \to 0$.

For $R > 0$ and $\varphi \in C_0^\infty([-R, R])$ we have,

$$
\begin{aligned}
< f_\epsilon, \varphi > &= \int_{\epsilon \leq |x| \leq R} \frac{\varphi(x)}{x} \,\mathrm{d}x \\
&= \int_{\epsilon \leq |x| \leq R} \frac{\varphi(x) - \varphi(0)}{x} \,\mathrm{d}x \\
&\leq (R - \epsilon) \max_{x \in [-R, R]} |\varphi'(x)| \\
&\leq R \max_{|x| \leq R} |\varphi(x)'|
\end{aligned}
$$

Hence, the integral is well defined and has a limit when $\epsilon \to 0$. We can define the following linear form,

$$
< \mathrm{vp}\left(\frac{1}{x}\right), \varphi > := \lim_{\epsilon \to 0} \int_{|x| \geq \epsilon} \frac{\varphi(x)}{x} \,\mathrm{d}x
$$

and by Theorem 1 it is a distribution.

Distributions Operations

We are going to extend to distributions several classical operations defined for functions. We are going to demand that the new definition is consistent with the old definition for functions.

Conjugate

If $u \in \mathcal{D}'(\Omega)$, we define the complex conjugate distribution $\bar{u}$ by,

$$\forall \varphi \in C_0^\infty(\Omega), \quad < \bar{u}, \varphi > := \overline{< u, \bar{\varphi} >}.$$

- Check that for $u \in L^1_{\text{loc}}$ this is the good definition.

We are going to say $u \in \mathcal{D}'(\Omega)$ is real iff $u = \bar{u}$. We can define in the same way $\mathrm{Re}(u)$ and $\mathrm{Im}(u)$.

Distributions Operations

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Translation

For a function $f : \mathbb{R}^n \to \mathbb{C}$ we now that $\tau_a f = f(x-a)$ is the translation of f of vector $a \in \mathbb{R}^n$. Since,

$$\int f(x-a)\varphi(x)\,\mathrm{d}x = \int f(x)\varphi(x+a)\,\mathrm{d}x$$

It is natural to define the translation of a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ as follows

$$\forall \varphi \in C_0^\infty(\mathbb{R}^n), \quad < \tau_a u, \varphi > := < u, \tau_{-a}\varphi > .$$

Distributions Operations

We are going to extend to distributions several classical operations defined for functions. We are going to demand that the new definition is consistent with the old definition for functions.

Dilatation

For a function $f : \mathbb{R}^n \to \mathbb{C}$ we now that $f_\lambda(x) = f(\frac{x}{\lambda})$ is the dilatation of f a scale $\lambda \in \mathbb{R}^n$. Since,

$$\int f\left(\frac{x}{\lambda}\right)\varphi(x)\,\mathrm{d}x = \int f(x)\varphi(\lambda x)|\lambda|^n\,\mathrm{d}x$$

It is natural to define the dilatation of a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ as follows

$$\forall \varphi \in C_0^\infty(\mathbb{R}^n), \quad < u_\lambda, \varphi > := |\lambda|^n < u, \varphi_{\frac{1}{\lambda}} > .$$

Distribution Operations

Product of Distributions

In general, it is impossible to define the product of two distributions that extends, in a continuous way, the product of functions. For example

$$f_j(x) = \begin{cases} j & \text{if } |x| \leq [-\frac{1}{2j}, \frac{1}{2j}], \\ 0 & \text{otherwise.} \end{cases}$$

We know that $f_j \xrightarrow{\mathcal{D}'} \delta_0$ but f_j^2 doesn't have a limit.

However, we can define in a natural way, the product of a Distribution and a function in $\mathbb{C}^\infty$:

Theorem 2: Let $u \in \mathcal{D}'(\Omega)$ and $f \in C^\infty$. We define the product distribution $fu \in \mathcal{D}'(\Omega)$ by

$$< fu, \varphi > := < u, f\varphi > .$$

We need to prove that the definition is a distribution. It is a linear form, so we rest to prove that $\forall K \subset \Omega$, $\exists C, p$ such that

$$| < fu, \varphi > | \leq C \sup_{x \in K, |\mathbf{i}| \leq p} |\partial^{\mathbf{i}} \varphi|, \quad \forall \varphi \in C_K^\infty \quad \boxed{\text{TD Exercise!}}$$

Distributions Differentiation

We have defined the derivative of a distribution. We are going to show how we can calculate the derivative of a function that is not of class C^1.

Def 3: We say that a function f is in *C^m piecewise* if there exists a discrete number of points (a_i) such that the function f has continuous m-derivatives inside (a_i, a_{i+1}) and the derivatives can be continuously extended to the intervals $[a_i, a_{i+1}]$.

Notation: We write $f^{(k)}(a_i^+)$ and $f^{(k)}(a_i^-)$ the right and left limits of $f^{(k)}(x)$ at point a_i.

Theorem 3: Jumps Formula (*Formule des sauts*)
Let $f \in C^1$ piecewise in $I \subset \mathbb{R}$ with (a_i) the *discrete* number of discontinuities. Then

$$f' = \{f\}' + \sum_i \left(f(a_i^+) - f(a_i^-)\right) \delta_{a_i}$$

where

- f' is the derivative in the distributions sense
- $\{f\}'$ is the pointwise derivative of f.

Proof.

Take $\varphi \in C_c^\infty$ and integrate by parts $< f', \varphi >= -\int f \varphi'$. □

Notice that since φ has compact support, when applying f' to a test function only a finite number of jumps are summed, so everything is well defined.

Theorem 4: Let I be an open interval. If $u \in \mathcal{D}'(I)$ such that $u' = 0$, then $u = C$ a.e.

Proof.

First notice, that $\varphi \in C_c^\infty(I)$ has a primitive in $C_c^\infty(I)$ if and only if $\int \varphi(x) = 0$. The only primitive that is zero at the left of the support of φ is $\int_{-\infty}^{x} \varphi(t)dt$ and in order to be zero at right of the support of φ the integral should be zero.

Let $\theta \in C_c^\infty(I)$ such that $\int \theta(x)dx = 1$. Then, $\varphi - \theta \int \varphi(x)dx$ has zero integral and therefore there exists a unique $\psi \in C_c^\infty(I)$ such that

$$\varphi = \left(\int \varphi(x) \, \mathrm{d}x \right) \theta + \frac{\mathrm{d}\psi}{dx}$$

Next, take u verifying $u' = 0$. We have

$$< u, \varphi >= (\int \varphi(x) \, \mathrm{d}x) < u, \theta > + < u, \psi' > .$$

The last term is $< u, \psi' >= - < u', \psi >= 0$, so calling C to the constant $< u, \theta >$, we have

$$< u, \varphi >= C \int \varphi(x)dx, \quad \text{so } u = C \text{ a.e..}$$

$\square$