# Hilbert and Fourier analysis
# C8

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# Today's topics

- Distributions Theory
  - Summary from previous lecture
  - Periodic Sobolev Spaces
  - Poisson, Laplace Equations
  - Lax-Milgram Lemma

# Motivation...

Assume we are given a function $u \in C^k(\mathbb{R})$ and $\varphi \in C_c^\infty$ an auxiliary function that we will call "test function".

Let $u^{(k)}$ be the kth-derivative of $u$, then from the integration by parts

$$\int u^{(k)}(x)\varphi(x)\,\mathrm{d}x = (-1)^k \int u(x)\varphi(x)^{(k)}\,\mathrm{d}x$$

There are no boundary terms, since every "test function" $\varphi$ has compact support in $\mathbb{R}$.

Now, If $u \notin C^k$ but $u \in L^1_{\text{loc}}(\mathbb{R})$ the right side still makes sense.

| How can we use this to generalize the *derivative* for $u \in L^1_{\text{loc}}(\mathbb{R})$? |
|---|

We can define $u^{(k)}$ by means of the application to every *test function*

$$l_{u^{(k)}} : \varphi \longrightarrow (-1)^k \int u(x)\varphi(x)^{(k)}\,\mathrm{d}x$$

$l_{u^{(k)}}$ is not a function but is a linear form (we will call them *generalized functions* or *distributions*).

| We can define *something* by the application to a set of *test functions*! |
|---|

Def : We call distribution on $\Omega$ (open set in $\mathbb{R}^n$) to every linear form $u$ over $\mathcal{C}_c^\infty(\Omega)$ satisfying the following continuity property: for every compact set $K \subset \Omega$ there exists an integer $p = p(K)$ and a constant $C = C(K)$ such that for every function $\varphi \in \mathcal{C}_c^\infty(\Omega)$ with support included in $K$, we have

$$| < u, \varphi > | \leq C \sup_{|i| \leq p, x \in K} |\partial^i \varphi(x)|.$$

- We call $\mathcal{D}(\Omega) := C_c^\infty(\Omega)$ (test functions)
- We will refer to the vectorial space of Distributions as $\mathcal{D}'(\Omega)$ the *dual* of $C_c^\infty(\Omega)$.
- If **p** can be chosen independently of **K**, we say that the distribution is of finite order. The smallest value **p** is called the order of **u**.
- $u$ is continuous: Given a sequence of test functions $(\varphi_j)$ with same support $K$ that converges uniformly and also all their derivatives, then

$$< u, \varphi_j > \longrightarrow < u, \varphi > .$$

# Distributions: Examples

If $u \in L^1_{\text{loc}}(\Omega)$ then

$$< \tilde{u}, \varphi >:= \int_\Omega u(x)\varphi(x)\,\mathrm{d}x$$

Is this a Distribution? It is a linear form

and for every $K \subset \Omega$ compact and for $\varphi \in \mathcal{C}^\infty_K$, we have

$$| < \tilde{u}, \varphi > | \leq \underbrace{\left(\int_K |u(x)|\,\mathrm{d}x\right)}_{=C} \left(\sup_{x \in K} |\varphi(x)|\right).$$

it is a distribution of order 0 ($p$ independent of $K$, $p = 0$).

If two distributions $\tilde{u}$ and $\tilde{v}$ in $\mathbb{R}^n$ associated to functions $u$ and $v$ in $L^1_{\text{loc}}$, are equal ($\tilde{u} = \tilde{v}$) then $u = v$ *a.e.*

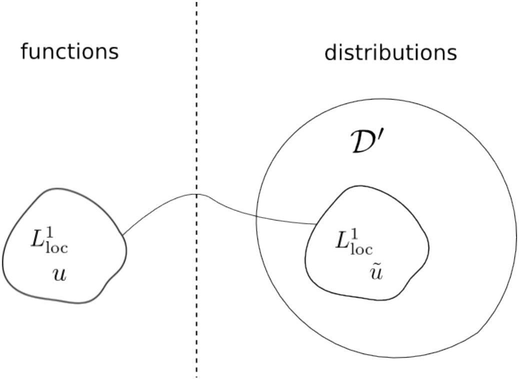
Thanks to this unicity result,

$$\tilde{u} = \tilde{v} \text{ in } \mathcal{D}'(\Omega) \Longleftrightarrow u = v \text{ in } L^1_{\text{loc}}(\Omega)$$

we can consider the functions $u \in L^1_{\text{loc}}$ as the distributions ($u$ and $\tilde{u}$).

$L^1, L^2, L^\infty \subset L^1_{\text{loc}}$

$L^\infty_{\text{loc}} \subset L^2_{\text{loc}} \subset L^1_{\text{loc}}$



# Dirac delta *function*

*Dirac* distribution in $x \in \mathbb{R}^n$ is defined by

$$< \delta_x, \varphi >= \varphi(x), \quad \forall \varphi \in C_c^\infty(\Omega)$$

*Dirac comb* (peigne de Dirac):

$$u = \sum_{\mathbf{n} \in \mathbb{Z}^n} \delta_{2\mathbf{n}\pi}$$

# Convergence of Distributions

Def 2: Let $(u_n)$ be a sequence of distributions on $\Omega$ and $u$ one distribution on $\Omega$. We say that $u_n$ converges to $u$ in the *distribution sense* if for every $\varphi \in \mathcal{C}_c^\infty(\Omega)$ the numerical sequence $(< u_n, \varphi >)_n$ converges to $< u, \varphi >$. We write

$$u_n \overset{\mathcal{D}'(\Omega)}{\longrightarrow} u$$

Theorem 1: If $(u_n)$ is a sequence of distributions on $\Omega$ such that $\forall \varphi \in \mathcal{C}_c^\infty$ the sequence $< u_n, \varphi >$ converges to a number, that we note $< u, \varphi >$ then, the limit $u$, is also a distribution in $\mathcal{D}'(\Omega)$.

# Convergence of Distributions

**Exercise** If $u_n \in L^1_{\text{loc}}$ converges to $u \in L^1_{\text{loc}}$, then $u_n$ also converges to $u$ in the distribution sense.

**Recall**: For sequences in $L^2_{\text{loc}}$ (or $L^\infty_{\text{loc}}$) the convergence in $L^2$ (or $L^\infty$) for every compact, implies the convergence in $L^1$ for every compact and therefore the convergence in the distribution sense.

# Derivative of Distributions

<u>Def :</u> Let $u \in \mathcal{D}'(\Omega)$, we call *partial derivative* of order $\mathbf{i}$ of $u$ in $\Omega$ to the distribution $\partial^{\mathbf{i}} u \in \mathcal{D}'(\Omega)$ defined by

$$< \partial^{\mathbf{i}} u, \varphi > := (-1)^{|\mathbf{i}|} < u, \partial^{\mathbf{i}} \varphi > .$$

- We can always differentiate a distribution $u \in \mathcal{D}'(\Omega)$.
- If $u \in C^k(\Omega)$ then $\partial^{\mathbf{j}} u$ with $|\mathbf{j}| \leq k$ is a function and a distribution!
- If $u \notin C^k(\Omega)$ then $\partial^{\mathbf{j}} u$ is not a function but it is a distribution!
- We need to learn how to calculate derivatives in the distribution sense when functions are not differentiable.
- It is a continuous operation! In general this is false in the case of functions.

Theorem 2: (*Derivative of a limit distribution*)
Let $u_j \in \mathcal{D}'(\Omega)$ be a sequence of distributions such that $u_j \stackrel{\mathcal{D}'(\Omega)}{\longrightarrow} u$, then the sequence $\partial^{\mathbf{i}} u_j \stackrel{\mathcal{D}'(\Omega)}{\longrightarrow} \partial^{\mathbf{i}} u$

Remark: Differentiation is always possible and moreover it is a continuous operation! In general this is false in the case of functions.

# Distributions Operations

We can extend to distributions several classical operations defined for functions. We demand that the new definition is consistent with the old definition for functions.

- Conjugate $< \bar{u}, \varphi > := \overline{< u, \bar{\varphi} >}$
- Translation $< \tau_a u, \varphi > := < u, \tau_{-a}\varphi >$
- Dilatation $< u_\lambda, \varphi > := |\lambda|^n < u, \varphi_{\frac{1}{\lambda}} >$

Although it is not possible to define the product between two distribution we can define

- Product between a function $f \in C^\infty$ and a distribution $u$, $< fu, \varphi > := < u, f\varphi >$.

Theorem : Jumps Formula (*Formule des sauts*)
Let $f \in C^1$ piecewise in $I \subset \mathbb{R}$ with $(a_i)$ the *discrete* number of discontinuities. Then

$$f' = \{f\}' + \sum_i \left(f(a_i^+) - f(a_i^-)\right) \delta_{a_i}$$

where

- $f'$ is the derivative in the distributions sense
- $\{f\}'$ is the pointwise derivative of $f$.

Proposition 1: Let $f$ defined in $\mathbb{R}$, $2\pi-$periodic and such that $f \in L^2(0, 2\pi)$. We can differentiate the Fourier series of $f$ any number of times, and we have

$$f^{(p)}(x) = \sum_{k=-\infty}^{\infty} (ik)^p c_k(f) \mathrm{e}^{ikx}$$

where the equality holds in the distribution sense in $\mathbb{R}$.

Remark: The equality is also valid as an equality between two distributions in $(0, 2\pi)$ or any other open interval of $\mathbb{R}$.

## Proof.

If $f \in L^2(0, 2\pi)$, $2\pi$-periodic then

(a) $f \in L^1_{\text{loc}}$ and so $f$ is a Distribution in $\mathcal{D}'(0, 2\pi)$.

(b) $f$ is the sum in the $L^2$ sense of its Fourier series, $f(x) = \sum_{k=-\infty}^{\infty} c_k(f)\mathrm{e}^{ikx}$ in $L^2_{\text{loc}}$

Consider the partial sums $s_n(f)$, we have

$$s_n(f) = \sum_{k=-n}^{n} c_k(f)\mathrm{e}^{ikx} \xrightarrow{L^2_{\text{loc}}} f.$$

Then, $s_n(f) \overset{\mathcal{D}'(0,2\pi)}{\longrightarrow} f$ and also in $\mathcal{D}'(\mathbb{R})$. From the Theorem of differentiation of a limit distribution, we have

$$s_n(f)^{(p)} \overset{\mathcal{D}'(\mathbb{R})}{\longrightarrow} f^{(p)}$$

or,

$$\sum_{k=-n}^{n} (ik)^p c_k(f)\mathrm{e}^{ikx} \overset{\mathcal{D}'(\mathbb{R})}{\longrightarrow} f^{(p)}(x)$$

Corollary 1: Let $f$ defined in $\mathbb{R}^2$, $2\pi \times 2\pi$ periodic and such that $f \in L^2((0,2\pi)^2)$. We can differentiate the Fourier series of $f$ any number of times, and we have

$$\frac{\partial^{p+q} f}{\partial x^p \partial y^q}(x,y) = \sum_{m,n=-\infty}^{\infty} (im)^p (in)^q c_{m,n}(f) \mathrm{e}^{i(mx+ny)}$$

where the equality holds in the distribution sense in $\mathbb{R}$.

# Poisson Summation Formula

We call Dirac Comb (peigne de Dirac) to the $2\pi$-periodization of the Dirac delta, defined by

$$\tilde{\delta} = \sum_{n \in \mathbb{Z}} \delta_{2n\pi}$$

Remark: $< \tilde{\delta}, \varphi >= \sum_{n \in \mathbb{Z}} \varphi(2n\pi), \quad \forall \varphi \in \mathcal{D}(\mathbb{R})$

Proposition 2: *Poisson Summation Formula*.

Let $s$ be the sawtooth wave (*dents de scie*), $2\pi$ periodic, defined in $[0, 2\pi)$ by $s(x) = \pi - x$. Then $s' = 2\pi\tilde{\delta} - 1$ and

$$\tilde{\delta} = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \mathrm{e}^{inx}$$

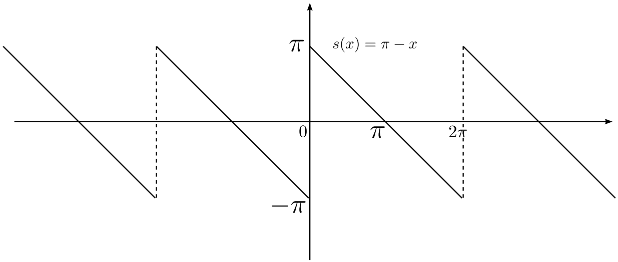
and therefore for $\varphi \in C_c^\infty$ we have the Poisson summation formula,

$$\sum_{k=-\infty}^{\infty} \varphi(2k\pi) = \frac{1}{2\pi} \sum_{p=-\infty}^{\infty} \tilde{\varphi}(p)$$

where we have called $\tilde{\varphi}(p) = \int_{\mathbb{R}} \varphi(x) \mathrm{e}^{-ipx} \, \mathrm{d}x$.

# Poisson Summation Formula

Let $s$ be the sawtooth wave (*dents de scie*), $2\pi$ periodic, defined in $[0, 2\pi)$ by $s(x) = \pi - x$. Then $s' = 2\pi\tilde{\delta} - 1$ and ... (go to the previous slide)



# Poisson Summation Formula

**Proof** Consider the Fourier Series development of function $s(x)$,

$$s(x) = \sum_{n=-\infty}^{\infty} c_n(s)\mathrm{e}^{inx} \quad \text{in } L^2(\mathbb{R})$$

with $c_n(s) = \frac{1}{2\pi}\int_0^{2\pi}(\pi - x)\mathrm{e}^{-inx}\,\mathrm{d}x = \begin{cases} \frac{1}{in} & \text{if } n \neq 0, \\ 0 & \text{if } n = 0. \end{cases}$

Then, we can write

$$s = \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{in}\mathrm{e}^{inx}.$$

The convergence is in $L^2_{\text{loc}}$, then also in $L^1_{\text{loc}}$ and therefore it is also in $\mathcal{D}'(\Omega)$. We can differentiate this expression in the distributions sense,

$$s'(x) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \mathrm{e}^{inx}.$$

From the other part, from the *Jumps Formula*, we have

$$s'(x) = -1 + (2\pi)\sum_{n=-\infty}^{\infty} \delta_{2n\pi}$$

# Poisson Summation Formula

**Proof (cont.)**

$$(2\pi)\sum_{n=-\infty}^{\infty}\delta_{2n\pi} = 1 + \sum_{n\in\mathbb{Z}\setminus\{0\}} \mathrm{e}^{inx} = \sum_{n\in\mathbb{Z}} \mathrm{e}^{inx}$$

Then we get the desired formula,

$$\sum_{n\in\mathbb{Z}}\delta_{2\pi n} = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \mathrm{e}^{inx}$$

If we apply this last expression to $\varphi \in C_c^{\infty}$ we get,

$$\sum_{k=-\infty}^{\infty}\varphi(2k\pi) = \frac{1}{2\pi}\sum_{p=-\infty}^{\infty}\tilde{\varphi}(p)$$

where we have called $\tilde{\varphi}(p) = \int_{\mathbb{R}}\varphi(x)\mathrm{e}^{-ipx}\,\mathrm{d}x$.

□

Assume we are given a function $u \in C^k(\mathbb{R})$ and $\varphi \in C_c^\infty$ an auxiliary function that we will call "test function".

Let $u^{(k)}$ be the kth-derivative of $u$, then from the integration by parts

$$\int u^{(k)}(x)\varphi(x)\,\mathrm{d}x = (-1)^k \int u(x)\varphi(x)^{(k)}\,\mathrm{d}x$$

There are no boundary terms, since every "test function" $\varphi$ has compact support in $\mathbb{R}$.

Now, If $u \notin C^k$ but $u \in L^1_{\text{loc}}(\mathbb{R})$ the right side still makes sense.

| How can we use this to generalize the *derivative* for $u \in L^1_{\text{loc}}(\mathbb{R})$? |
| --- |

If there exists $v \in L^1_{\text{loc}}(\mathbb{R})$ such that for every *test function*

$$\int v(x)\varphi(x)\,\mathrm{d}x = (-1)^k \int u(x)\varphi(x)^{(k)}\,\mathrm{d}x$$

then $v$ is called the *weak derivative* of $u$.

**Weak derivative** introduced by S. L. Sobolev 1930's!

Schwartz Theory of Distributions generalizes this idea (Dirac, ...).

# Sergei L'vovich Sobolev

Sobolev introduced the notions that are now fundamental for several different areas of mathematics. Sobolev spaces can be defined by some growth conditions on the Fourier transforms; they and their embedding theorems are an important subject in functional analysis. Generalized functions (later known as distributions) were first introduced by Sobolev in 1935 for weak solutions, and further developed by Laurent Schwartz. Sobolev abstracted the classical notion of differentiation so expanding the ranges of applications of the technique of Newton and Leibniz. The theory of distribution is considered now as the calculus of the modern epoch.

**Sergei L. Sobolev** (6 October 1908 - 3 January 1989 in Moscow)

# Periodic Sobolev Spaces: $H^m_{\text{per}}(0, 2\pi)$

Def 1: We call Periodic Sobolev Space of order $m \geq 1$ and we note by $H^m_{\text{per}}(0, 2\pi)$ to the set of functions $u$ such that:

- $u \in L^2(0, 2\pi)$ and $u$ is $2\pi$-periodic in $\mathbb{R}$
- $u^{(i)} \in L^2(0, 2\pi)$, for $i \leq m$.

The derivative is in the *distribution sense*.

Def 2: We provide $H^m_{\text{per}}$ with the following norm:

$$\|u\|_{H^m_{\text{per}}} = \left( \sum_{i \leq m} \|u^{(i)}\|^2_{L^2} \right)^{\frac{1}{2}}$$

associated to the hermitian product:

$$(u, v)_{H^m_{\text{per}}} = \sum_{n \leq m} \int_0^{2\pi} u^{(n)}(x) \overline{v^{(n)}(x)} dx$$

**Exercise:** Show that $\| \cdot \|_{H^m_{\text{per}}}$ is a norm for $H^m_{\text{per}}$ and $(\cdot, \cdot)_{H^m_{\text{per}}}$ is a hermitian product.

Remark: The $H^m_{\text{per}}(0, 2\pi)$ spaces are simple to analyze as we can characterize them by using Fourier series.

Proposition 1: If $u \in H^m_{\text{per}}$, then for every $n \leq m$, the Fourier coefficients of $u^{(n)}$ satisfy

$$c_k(u^{(n)}) = (ik)^n c_k(u).$$

## Proof.

Since $u \in L^2(0, 2\pi)$ we can write

$$u(x) = \sum_k c_k(u)\mathrm{e}^{ikx}$$

where the equality holds in the $L^2_{\text{loc}}(\mathbb{R})$ provided we periodize $u$.

From pervious lecture if $u \in L^2(0, 2\pi)$, $2\pi$-periodic, it can be differentiated any number of times by its Fourier series :

$$u^{(n)}(x) = \sum_k c_k(u)(ik)^n \mathrm{e}^{ikx} \quad \text{convergence in } \mathcal{D}'(\mathbb{R}).$$

Since we have supposed $u^{(n)} \in L^2(0, 2\pi)$, then it can be developed by its Fourier Series

$$u^{(n)}(x) = \sum_k c_k(u^{(n)})\mathrm{e}^{ikx}$$

with $(c_k(u^{(n)}))_k \in l^2(\mathbb{Z})$. Then by the unicity of the Fourier Series expansion of a distribution (*will be proved next lecture*) we have

$$c_k(u)(ik)^n = c_k(u^{(n)}).$$

Proposition 2: The space $H^m_{\mathrm{per}}(0, 2\pi) \subset C^{m-1}(\mathbb{R})$. In particular, the functions of $H^1_{\mathrm{per}}(0, 2\pi)$ are continuous.

Proof.

For $m = 1$.

Then the Fourier series of $u$ satisfies

$$c_k(u) = \frac{1}{ik} c_k(u') \in l^1(\mathbb{Z})$$

since $\frac{1}{ik} \in l^2(\mathbb{Z})$ and $c_k(u') \in l^2(\mathbb{Z})$. Hence the series

$$\sum_k c_k(u) \mathrm{e}^{ikx} \quad \text{converges uniformly in } \mathbb{R}.$$

Finally, the limit is continuos ($u$ is equal to a continuous function).

For $m > 1$ proceed by recurrence:

if $u \in H^m_{\mathrm{per}}(0, 2\pi)$, then $u^{(m-1)} \in H^1_{\mathrm{per}}(0, 2\pi)$.

□

**Proposition 3:** The spaces $H^m_{\text{per}}(0, 2\pi)$ are Hilbert spaces and their norm can be written

$$\|u\|^2_{H^m_{\text{per}}} = \sum_{n \leq m} \|u^{(n)}\|^2_{L^2} = \sum_k |c_k(u)|^2 (1 + |k|^2 + \ldots + |k|^{2m}).$$

An equivalent Hilbert norm is

$$|u|^2_{H^m_{\text{per}}} = \sum_k |c_k(u)|^2 (1 + |k|^{2m}).$$

## Proof.

Consider the application $\psi : H^m_{\text{per}}(0, 2\pi) \longrightarrow l^2(\mathbb{Z})$ defined by

$$\psi(u) = \left( c_k(u)(1 + |k|^2 + \ldots + |k|^{2m})^{\frac{1}{2}} \right)_k .$$

From Parseval $\psi$ is an isometry from $H^m_{\text{per}}$ to $l^2(\mathbb{Z})$. Since $l^2(\mathbb{Z})$ is a Hilbert space then $H^m_{\text{per}}(0, 2\pi)$ is also a Hilbert space.

The equivalence between norms is immediate. □

## Other Fourier Bases

Let us consider functions in in $L^2(0, 2\pi)$. We know

$$f(x) = \sum_{k\in\mathbb{Z}} c_k(f)\mathrm{e}^{ikx} \quad \text{in } L^2\text{sense.}$$

Moreover, $\left(\frac{1}{\sqrt{2\pi}}\mathrm{e}^{ikx}\right)_k$ is a Hilbert basis of $L^2(0, 2\pi)$.

The following are also Hilbert bases of $L^2(0, 2\pi)$ :

- $\frac{1}{\sqrt{2\pi}},\ \frac{1}{\sqrt{\pi}}\cos(kx),\ \frac{1}{\sqrt{\pi}}\sin(kx) \quad k = 1, 2, \ldots$
- $\frac{1}{\sqrt{2\pi}},\ \frac{1}{\sqrt{\pi}}\cos(\frac{k}{2}x), \quad k = 1, 2, \ldots$ (Cosine basis)
- $\frac{1}{\sqrt{\pi}}\sin(\frac{k}{2}x), \quad k = 1, 2, \ldots$ (Sine basis)

Theorem 1: *Problème de l'élastique chargé*

$$(P) \quad \begin{cases} -u'' = f, & f \in L^2(0, 2\pi) \\ u(0) = u(2\pi) = 0. \end{cases}$$

Then, there exists a unique $u \in L^2(0, 2\pi)$ solution of (P) in the distribution sense.

This solution belongs to $H^2_{\mathrm{per}}(-2\pi, 2\pi)$ ($4\pi$-periodic functions).

**Proof**

We need $u(0) = u(2\pi) = 0$, it may be convenient to decompose $f$ and $u$ in the sine basis: $\left(\sin\left(\frac{kx}{2}\right)\right)_{k\in\mathbb{N}^*}$. (implies extending $f$ to an odd function over $[-2\pi, 2\pi]$ and computing its Fourier Series).

Then, $f(x) = \sum_{k\in\mathbb{N}^*} c_k(f) \sin\frac{kx}{2}$ where $c_k(f) = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin\frac{kt}{2}dt$.

Also, $u(x) = \sum_{k\in\mathbb{N}^*} c_k(u)\sin\frac{kx}{2}$ and $u''(x) = -\sum_{k\in\mathbb{N}^*} c_k(u)\frac{k^2}{4}\sin\frac{kx}{2}$.

Hence, the main equation in Problem (P) can be re-written as

$$\sum_{k\in\mathbb{N}^*} c_k(u)\frac{k^2}{4}\sin\frac{kx}{2} = \sum_{k\in\mathbb{N}^*} c_k(f)\sin\frac{kx}{2}$$

From the unicity of the Fourier coefficients for a distribution (that causes the unicity of representation in the sine basis) we have

$$\frac{k^2}{4}c_k(u) = c_k(f), \forall k \in \mathbb{N}^* \iff c_k(u) = \frac{4}{k^2}c_k(f), \forall k \in \mathbb{N}^*$$

Then, $u = 4\sum_{k\in\mathbb{N}^*} \frac{c_k(f)}{k^2}\sin\frac{k}{2}x$ and $u \in H^2_{\text{per}}(-2\pi, 2\pi)$.

The function $u$ is not necessary $C^2$ so the problem $(P)$ doesn't make sense in the classical way!

As $u \in H^2_{\text{per}}(-2\pi, 2\pi)$, $u \in C^1$ and it's an odd function so $u(0) = u(2\pi) = 0$. □

# Periodic Sobolev Spaces in dimension 2: $H^m_{\text{per}}([0, 2\pi]^2)$

Def 3: We call Periodic Sobolev Space of order $m \geq 1$ and we note by $H^m_{\text{per}}([0, 2\pi]^2)$ to the set of functions $u$ such that:

- $u \in L^2([0, 2\pi]^2)$; $2\pi$-periodic over $\mathbb{R}^2$
- $\partial^{\mathbf{i}} u \in L^2([0, 2\pi]^2)$, for $|i| \leq m$.

The derivative is in the *distribution sense*.

Def 4: We provide $H^m_{\text{per}}([0, 2\pi]^2)$ with the following norm:

$$\|u\|_{H^m_{\text{per}}} = \left( \sum_{|\mathbf{i}| \leq m} \|\partial^{\mathbf{i}} u\|^2_{L^2} \right)^{\frac{1}{2}}$$

associated to the hermitian product:

$$(u, v)_{H^m_{\text{per}}} = \int_{[0,2\pi]^2} \sum_{|\mathbf{i}| \leq m} \partial^{\mathbf{i}} u(\mathbf{x}) \partial^{\mathbf{i}} \overline{v}(\mathbf{x}) d\mathbf{x}$$

Proposition 4: The spaces $H^m_{\text{per}}([0,2\pi]^2)$ are Hilbert spaces. An equivalent Hilbert norm is

$$\|u\|^2_{H^m_{\text{per}}} = \sum_{\mathbf{n}\in\mathbb{Z}^2} |c_{\mathbf{n}}(u)|^2(1+\|\mathbf{n}\|^2)^m.$$

The proof is identical to the one for dimension 1.

**Proposition 5:** The space $H^m_{\text{per}}([0,2\pi]^2) \subset C^{m-2}(\mathbb{R}^2)$. In particular, the functions of $H^2_{\text{per}}([0,2\pi]^2)$ are continuous.

## Proof.

For $m = 2$.

Let us take $u \in H^2_{\text{per}}([0,2\pi]^2)$. We are going to show that $u$ is continuous.

We define $d_{\mathbf{k}} = c_{\mathbf{k}}(u)(1 + \|\mathbf{k}\|^2)$. Next, $c_{\mathbf{k}} = \frac{d_{\mathbf{k}}}{1+\|\mathbf{k}\|^2}$.
Since $u \in H^2_{\text{per}}([0,2\pi]^2)$ then $d_{\mathbf{k}} \in l^2(\mathbb{Z}^2)$. Also, $\frac{1}{1+\|\mathbf{k}\|^2} \in l^2(\mathbb{Z}^2)$.

Therefore $c_{\mathbf{k}}(u) \in l^1(\mathbb{Z}^2)$ and the Fourier series of $u$

$$\sum_{\mathbf{k}} c_{\mathbf{k}}(u) \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} \quad \text{converges uniformly in } \mathbb{R}^2.$$

Finally, the limit is continuos ($u$ is equal to a continuous function).

For $m > 2$ proceed by recurrence:

if $u \in H^m_{\text{per}}([0,2\pi]^2)$, then $u^{(m-2)} \in H^2_{\text{per}}([0,2\pi]^2)$.

□

# Poisson Equation with Dirichlet conditions

We note the Laplacian of a function $u$ by $\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy}$.
Let be $\Omega = [0, 2\pi]^2$ and $\partial\Omega$ its border.

**Theorem 2:** *Equation de Poisson avec condition de Dirichlet*

- Let $f \in L^2([0, 2\pi]^2)$ (the *heat source*).
- Consider the Poisson equation in a square in isothermal conditions (temperature in the square border is fixed to 0). This is called *Poisson equation with Dirichlet condition*.
- The equation that represents the temperature inside the square can be written as

$$\begin{cases} -\Delta u = f, & \text{with } f \in L^2([0, 2\pi]^2) \quad (*) \\ u = 0 & \text{in } \partial\Omega. \end{cases}$$

Then, there exists a unique $u \in L^2([0, 2\pi]^2)$ solution (in the *distribution sense*). This solution is an odd (impaire) function in $H^2_{\text{per}}([-2\pi, 2\pi]^2)$ (a $4\pi$- periodic function).

**Proof** In order to naturally have the boundary condition $u = 0$ it is convenient to use the sine basis decomposition

$$\left(\sin\left(\tfrac{kx}{2}\right)\sin\left(\tfrac{ly}{2}\right)\right)_{k,l}, \text{ with } k,l \in \mathbb{N}^*$$

Hence, we write

$$u(x,y) = \textstyle\sum_{k,l\in\mathbb{N}^*} c_{k,l}(u)\sin\left(\tfrac{kx}{2}\right)\sin\left(\tfrac{ly}{2}\right)$$

$$f(x,y) = \textstyle\sum_{k,l\in\mathbb{N}^*} c_{k,l}(f)\sin\left(\tfrac{kx}{2}\right)\sin\left(\tfrac{ly}{2}\right)$$

The equation $(*)$ can be written

$$\sum_{k,l\in\mathbb{N}^*} c_{k,l}(f)\sin\left(\frac{kx}{2}\right)\sin\left(\frac{ly}{2}\right) = \sum_{k,l\in\mathbb{N}^*} c_{k,l}(u)\left(\frac{k^2+l^2}{4}\right)\sin\left(\frac{kx}{2}\right)\sin\left(\frac{ly}{2}\right)$$

By the unicity of the Fourier decomposition, we have

$$c_{k,l}(f) = c_{k,l}(u)\tfrac{k^2+l^2}{4}, \quad \forall k,l \in \mathbb{N}^*$$

Then

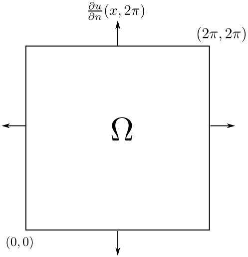
$$u(x,y) = 4\sum_{k,l\in\mathbb{N}^*} \frac{c_{k,l}(f)}{k^2+l^2}\sin\left(\frac{kx}{2}\right)\sin\left(\frac{ly}{2}\right)$$

Notice that, $u \in H^2_{\text{per}}([-2\pi, 2\pi]^2)$, $u \notin C^2$, but $u \in C^0$. Since $u$ is odd, then we have $u(x,y) = 0$ in $\partial\Omega$. □

# Normal derivative

Suppose $\Omega = [0, 2\pi]^2$. We call $\frac{\partial u}{\partial n}$ the normal derivative on the boundary of the domain to

$$
\begin{aligned}
\frac{\partial u}{\partial n}(x,0) &= -\frac{\partial u}{\partial y}(x,0)\\
\frac{\partial u}{\partial n}(x,2\pi) &= \frac{\partial u}{\partial y}(x,2\pi)\\
\frac{\partial u}{\partial n}(0,y) &= -\frac{\partial u}{\partial x}(0,y)\\
\frac{\partial u}{\partial n}(2\pi,y) &= \frac{\partial u}{\partial x}(2\pi,y).
\end{aligned}
$$




The easiest way to impose $\frac{\partial u}{\partial n} = 0$ is to consider even (paires) functions in the two variables, $4\pi$-periodic:

$$
\begin{aligned}
u(-x,y) &= u(x,y)\\
u(x,-y) &= u(x,y)
\end{aligned}
$$

for $(x, y) \in \Omega$, and since $u$ is $4\pi$-periodic, if $u \in C^1$ then, $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$.

**Exercise:** Prove it!

# Poisson Equation with Neumann conditions

Let be $\Omega = [0, 2\pi]^2$ and $\partial\Omega$ its border.

<u>Theorem 3:</u> *Equation de Poisson avec condition de Neumann*

- Let $f \in L^2(\Omega)$, such that $\int_\Omega f(x, y)dxdy = 0$ (the *heat source*).
- Consider the Poisson equation in a square with adiabatic boundary (e.g. heat flux though the boundary is zero). This is called *Poisson equation with Neumann condition*.
- The equation that represents the temperature inside the square can be written as

$$\begin{cases} -\Delta u = f, & \text{with } f \in L^2(\Omega),\ \int_\Omega f = 0 \quad (*) \\ \frac{\partial u}{\partial n} = 0 & \text{in } \partial\Omega. \end{cases}$$

Then, there exists $u \in L^2([0, 2\pi]^2)$ solution (in the *distribution sense*).

The solution satisfies $\frac{\partial u}{\partial n} = 0$ <u>in the sense</u> that is an even (paire) function.

This solution is unique up to a constant and belongs to $H^2_{\text{per}}([-2\pi, 2\pi]^2)$ (a $4\pi$- periodic function).

**Proof** In order to naturally have the boundary condition $\frac{\partial u}{\partial n} = 0$ it is convenient to use the cosine basis decomposition

$$\left(\cos\left(\tfrac{kx}{2}\right)\cos\left(\tfrac{ly}{2}\right)\right), \text{ with } k, l \in \mathbb{N}.$$

Hence, we write

$$u(x, y) = \sum_{k,l\in\mathbb{N}} c_{k,l}(u)\cos\left(\tfrac{kx}{2}\right)\cos\left(\tfrac{ly}{2}\right)$$

$$f(x, y) = \sum_{k,l\in\mathbb{N}} c_{k,l}(f)\cos\left(\tfrac{kx}{2}\right)\cos\left(\tfrac{ly}{2}\right)$$

with

$$c_{k,l} = c(k)c(l)\int_\Omega u(t, s)\cos\left(\tfrac{kt}{2}\right)\cos\left(\tfrac{ls}{2}\right) dtds$$

where $c(n) = \frac{1}{\pi}$ iff $n = 0$ or $c(n) = \frac{1}{2\pi}$ otherwise.

The equation $(*)$ can be written

$$\sum_{k,l\in\mathbb{N}} c_{k,l}(f)\cos\left(\frac{kx}{2}\right)\cos\left(\frac{ly}{2}\right) = \sum_{k,l\in\mathbb{N}} c_{k,l}(u)\left(\frac{k^2 + l^2}{4}\right)\cos\left(\frac{kx}{2}\right)\cos\left(\frac{ly}{2}\right).$$

By the unicity of the Fourier decomposition, we have

$$c_{k,l}(u)\frac{k^2+l^2}{4} = c_{k,l}(f) \quad \forall k, l \in \mathbb{N}.$$

Then,

$$c_{k,l}(u) = 4\frac{c_{k,l}(f)}{k^2+l^2}, \; (k, l) \neq (0, 0)$$

Recall that $\int f = 0$ and hence $c_{0,0}(f) = 0$.

**Proof (cont.)**

Next, we have

$$u(x,y) = c + 4 \sum_{k,l \in \mathbb{N}^*} \frac{c_{k,l}(f)}{k^2 + l^2} \cos\left(\frac{kx}{2}\right) \cos\left(\frac{ly}{2}\right)$$

where the constant $c$ is not determined by the system so there is one degree of freedom ($c_{0,0}(u)$ not defined).

- Notice that, $u \in H^2_{\text{per}}([-2\pi, 2\pi]^2)$.
- Remark that $u \notin C^2$, but $u \in C^0$ (the equation only makes sense in the distribution sense).
- Since $u$ is even (paire), we have $\frac{\partial u}{\partial n}(x,y) = 0$ in $\partial\Omega$. (note that the function may not be differentiable in the boundary, however if it is, the derivative must be zero).

$\square$

Theorem 4: Consider a bilinear form defined in a Hilbert space:

$$a(u, v) : H \times H \longrightarrow \mathbb{C}$$

Then, $a$ is continuous if and only if $\exists C > 0$ such that

$$|a(u, v)| \leq C\|u\|\|v\|, \forall u, v \in H.$$

**Proof**

($\Rightarrow$) If $a$ is continuos in 0, $\exists \eta > 0$ such that $|a(u, v)| \leq 1$ for $\|u\|, \|v\| \leq \eta$.

Consider $\eta \frac{u}{\|u\|}$ and $\eta \frac{v}{\|v\|}$ we deduce immediately: $|a(u, v)| \leq \frac{\|u\|\|v\|}{\eta^2}$.

($\Leftarrow$) From the linearity and the triangular inequality we have

$$\begin{aligned}
|a(u, v) - a(u', v')| &= |a(u + u' - u', v) - a(u', v' + v - v)| \\
&= |a(u - u', v) + a(u', v) - a(u', v' - v) - a(u', v)| \\
&\leq |a(u - u', v)| + |a(u', v' - v)| \\
&\leq C\|u - u'\|\|v\| + C\|u'\|\|v' - v\|
\end{aligned}$$

and we have the continuity. □

# Coercivity

We say that a symmetric bilinear form is coercive if there exists a constant $C > 0$ such that

$$a(u,u) \geq c\|u\|^2 \quad \forall u \in H$$

**Exercise**

Let $a(u,v)$ be a continuous and coercive bilinear form in $H$. Then

$$\|u\|_a := \sqrt{a(u,u)}$$

is an equivalent norm of the norm $\|\cdot\|$ in $H$.

(define the same topology / define the same convergence criterion).

**Remark**

A lot of problems in physics, mechanics, signal processing can be reduced to minimize in a Hilbert space $(H)$, an energy of the form

$$(*) \quad E(u) = \frac{1}{2}a(u,u) - (b,u)$$

where $b \in H$ and $a(.,.)$ is bilinear, continuous and coercive.

# Lax-Milgram Lemma

Theorem 5: Let $H$ be a Hilbert separable space and $E$ an energy functional of the type

$$(*) \quad E(u) = \frac{1}{2}a(u,u) - (b,u),$$

with $a(\cdot,\cdot)$ **bilinear**, **symmetric**[1], **continuous** and **coercive**.

Then, there exists a unique $u$, minimizer of $E(u)$ in $H$.

Moreover, $u$ is characterized by the following variational equation

$$\forall v \in H, \quad \mathrm{Re}(a(u,v)) = (b,v).$$

[1] $a(u,v) = \overline{a(v,u)}$

**Proof**

Assume the minimizer $u$ exists. Then, $\forall v \in H$ consider $f(t) = E(u + tv)$ where $t \in \mathbb{R}$.

$$f(t) = \frac{1}{2}a(u + tv, u + tv) - (b, u + tv)$$
$$= \frac{1}{2}\left(a(u,u) + ta(v,u) + ta(u,v) + t^2 a(v,v)\right) - (b,u) - t(b,v)$$

then

$$f'(t) = \frac{1}{2}a(v,u) + \frac{1}{2}a(u,v) + ta(v,v) - (b,v).$$

Since $u$ is minimizer, we necessary have $f'(0) = 0$, then

$$\frac{1}{2}a(v,u) + \frac{1}{2}a(u,v) - (b,v) = 0$$

and we get the desired expression

$$\mathrm{Re}(a(v,u)) = (b,v) \quad \text{for all } v \in H.$$

**Proof (cont. 1)** The minimum is unique.

If there are two $u$ and $v$ then by the expression we have just proved

$$\operatorname{Re}(a(u, v-u)) = (b, v-u) = \operatorname{Re}(a(v, v-u)).$$

Thus $\operatorname{Re}(a(v-u, v-u)) = 0$ implying $a(v-u, v-u) = 0$.

Since $a$ is coercive

$$0 = a(u-v, u-v) \geq C\|u-v\|^2$$

and then

$$u = v.$$

## 4.2 Convergence faible

**Définition 4.4** *On dit qu'une suite $u_n$ dans un Hilbert converge faiblement vers $u \in H$ si pour tout $v \in H$ on a $(v, u_n) \to (v, u)$. On écrit alors $u_n \rightharpoonup u$.*

**Proposition 4.3** *Si $u_n \in H$ séparable est bornée ($||u_n|| \leq C$), alors il existe une sous-suite $(u_{n_k})_k$ qui converge faiblement :*

$$\forall v \in H,\ (u_{n_k}, v) \to (u, v)$$

*et on a*

$$||u|| \leq \liminf_n ||u_n||.$$

## Proof (cont. 2)

The minimum exists.

First we prove that $E(u)$ is lower bounded. From the coercivity and Cauchy-Schwartz inequality,

$$E(u) = \frac{1}{2}a(u,u) - (b,u) \geq \frac{1}{2}C\|u\|^2 - (b,u) \geq \frac{1}{2}C\|u\|^2 - \|b\|\|u\|$$

which is a real function of $\|u\|$ lower bounded. Thus, we can consider a minimizing sequence $(u_n)_n \subset H$ for $E$ such that $E(u_n) \to \inf_{u \in H} E(u)$.

We have $E(u_n) \leq E(0) = 0$, and then by the coercivity

$$0 \geq E(u_n) \geq \frac{c}{2}\|u_n\|^2 - \|b\|\|u_n\|$$

next

$$\|u_n\| \leq \frac{2}{c}\|b\|$$

that is, the minimizing sequence is bounded in $H$.

From Proposition 4.3 there exits a subsequence (we also note it $u_n$) such that

$$u_n \rightharpoonup u \in H \quad \text{(weak convergence).}$$

## Proof (cont. 3)

By applying the same Proposition 4.3 with the equivalent norm $\|u\|_a$ we get,

$$\|u\|_a \leq \liminf_n \|u_n\|_a \quad \iff \quad a(u,u) \leq \liminf_n a(u_n,u_n)$$

Since there's weak convergence

$$\lim_n (b, u_n) = (b, u).$$

Next,

$$\begin{aligned} E(u) &= \frac{1}{2} a(u,u) - (b,u) \\ &\leq \liminf_n \frac{1}{2} a(u_n,u_n) - \lim_n (b,u_n) \\ &= \liminf_n E(u_n) \end{aligned}$$

Since $(u_n)_n$ is a minimizing sequence

$$\liminf_n E(u_n) = \inf_{v \in H} E(v)$$

Finally,

$$E(u) = \inf_{v \in H} E(v)$$

□