Hilbert and Fourier analysis
C9

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Today's topics

- Distributions Theory
  - Summary from previous lecture
  - The variational interpretation
  - Poisson editor
  - Unicity of Fourier expansion (distributions)
  - Functions in Sobolev spaces

Periodic Sobolev Spaces: $H^m_{\text{per}}(0, 2\pi)$

Def 1: We call Periodic Sobolev Space of order $m \geq 1$ and we note by $H^m_{\text{per}}(0, 2\pi)$ to the set of functions u such that:

- $u \in L^2(0, 2\pi)$ and u is 2π-periodic in $\mathbb{R}$
- $u^{(i)} \in L^2(0, 2\pi)$, for $i \leq m$.

The derivative is in the *distribution sense*.

Def 2: We provide H^m_{per} with the following norm:

$$\|u\|_{H^m_{\text{per}}} = \left(\sum_{i \leq m} \|u^{(i)}\|^2_{L^2} \right)^{\frac{1}{2}}$$

associated to the hermitian product:

$$(u, v)_{H^m_{\text{per}}} = \sum_{n \leq m} \int_0^{2\pi} u^{(n)}(x) \overline{v^{(n)}(x)} dx$$

Exercise: Show that $\| \cdot \|_{H^m_{\text{per}}}$ is a norm for H^m_{per} and $(\cdot, \cdot)_{H^m_{\text{per}}}$ is a hermitian product.

Remark: The $H^m_{\text{per}}(0, 2\pi)$ spaces are simple to analyze as we can characterize them by using Fourier series.

Proposition : If $u \in H^m_{\text{per}}$, then for every $n \leq m$, the Fourier coefficients of $u^{(n)}$ satisfy

$$c_k(u^{(n)}) = (ik)^n c_k(u).$$

Proposition : The space $H^m_{\text{per}}(0,2\pi) \subset C^{m-1}(\mathbb{R})$. In particular, the functions of $H^1_{\text{per}}(0,2\pi)$ are continuous.

Proposition : The spaces $H^m_{\text{per}}(0,2\pi)$ are Hilbert spaces and their norm can be written

$$\|u\|^2_{H^m_{\text{per}}} = \sum_{n\leq m} \|u^{(n)}\|^2_{L^2} = \sum_k |c_k(u)|^2(1+|k|^2+\ldots+|k|^{2m}).$$

An equivalent Hilbert norm is

$$|u|^2_{H^m_{\text{per}}} = \sum_k |c_k(u)|^2(1+|k|^{2m}).$$

Other Fourier Bases

Let us consider functions in in $L^2(0, 2\pi)$. We know

$$f(x) = \sum_{k\in\mathbb{Z}} c_k(f)\mathrm{e}^{ikx} \quad \text{in } L^2\text{sense.}$$

Moreover, $\left(\frac{1}{\sqrt{2\pi}}\mathrm{e}^{ikx}\right)_k$ is a Hilbert basis of $L^2(0, 2\pi)$.

The following are also Hilbert bases of $L^2(0, 2\pi)$:

- $\frac{1}{\sqrt{2\pi}}, \ \frac{1}{\sqrt{\pi}}\cos(kx), \ \frac{1}{\sqrt{\pi}}\sin(kx) \quad k = 1, 2, \dots$
- $\frac{1}{\sqrt{2\pi}}, \ \frac{1}{\sqrt{\pi}}\cos(\frac{k}{2}x), \quad k = 1, 2, \dots$ (Cosine basis)
- $\frac{1}{\sqrt{\pi}}\sin(\frac{k}{2}x), \quad k = 1, 2, \dots$ (Sine basis)

Theorem 1: *Problème de l'élastique chargé*

$$(P) \quad \begin{cases} -u'' = f, & f \in L^2(0, 2\pi) \\ u(0) = u(2\pi) = 0. \end{cases}$$

Then, there exists a unique $u \in L^2(0, 2\pi)$ solution of (P) in the distribution sense.

This solution belongs to $H^2_{\text{per}}(-2\pi, 2\pi)$ (4π-periodic functions).

Proof

We need $u(0) = u(2\pi) = 0$, it may be convenient to decompose $f$and u in the sine basis: $\left(\sin\left(\frac{kx}{2}\right)\right)_{k\in\mathbb{N}^*}$. (implies extending f to an odd function over $[-2\pi, 2\pi]$ and computing its Fourier Series).

Then, $f(x) = \sum_{k\in\mathbb{N}^*} c_k(f)\sin\frac{kx}{2}$ where $c_k(f) = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin\frac{kt}{2}dt$.

Also, $u(x) = \sum_{k\in\mathbb{N}^*} c_k(u)\sin\frac{kx}{2}$ and $u''(x) = -\sum_{k\in\mathbb{N}^*} c_k(u)\frac{k^2}{4}\sin\frac{kx}{2}$.

Hence, the main equation in Problem (P) can be re-written as

$$\sum_{k\in\mathbb{N}^*} c_k(u)\frac{k^2}{4}\sin\frac{kx}{2} = \sum_{k\in\mathbb{N}^*} c_k(f)\sin\frac{kx}{2}$$

From the unicity of the Fourier coefficients for a distribution (that causes the unicity of representation in the sine basis) we have

$$\frac{k^2}{4}c_k(u) = c_k(f), \forall k \in \mathbb{N}^* \iff c_k(u) = \frac{4}{k^2}c_k(f), \forall k \in \mathbb{N}^*$$

Then, $u = 4\sum_{k\in\mathbb{N}^*} \frac{c_k(f)}{k^2}\sin\frac{k}{2}x$ and $u \in H^2_{\text{per}}(-2\pi, 2\pi)$.

The function u is not necessary C^2 so the problem (P) doesn't make sense in the classical way!

As $u \in H^2_{\text{per}}(-2\pi, 2\pi)$, $u \in C^1$ and it's an odd function so $u(0) = u(2\pi) = 0$. □

Periodic Sobolev Spaces in dimension 2: $H^m_{\text{per}}([0,2\pi]^2)$

Def 3: We call Periodic Sobolev Space of order $m \geq 1$ and we note by $H^m_{\text{per}}([0,2\pi]^2)$ to the set of functions u such that:

- $u \in L^2([0,2\pi]^2)$; 2π-periodic over $\mathbb{R}^2$
- $\partial^{\mathbf{i}} u \in L^2([0,2\pi]^2)$, for $|i| \leq m$.

The derivative is in the *distribution sense*.

Def 4: We provide $H^m_{\text{per}}([0,2\pi]^2)$ with the following norm:

$$\|u\|_{H^m_{\text{per}}} = \left(\sum_{|\mathbf{i}| \leq m} \|\partial^{\mathbf{i}} u\|^2_{L^2} \right)^{\frac{1}{2}}$$

associated to the hermitian product:

$$(u,v)_{H^m_{\text{per}}} = \int_{[0,2\pi]^2} \sum_{|\mathbf{i}| \leq m} \partial^{\mathbf{i}} u(\mathbf{x}) \partial^{\mathbf{i}} \overline{v}(\mathbf{x}) d\mathbf{x}$$

Proposition : The space $H^m_{\text{per}}([0,2\pi]^2) \subset C^{m-2}(\mathbb{R}^2)$. In particular, the functions of $H^2_{\text{per}}([0,2\pi]^2)$ are continuous.

Proposition : The spaces $H^m_{\text{per}}([0,2\pi]^2)$ are Hilbert spaces. An equivalent Hilbert norm is

$$\|u\|^2_{H^m_{\text{per}}} = \sum_{\mathbf{n}\in\mathbb{Z}^2} |c_{\mathbf{n}}(u)|^2 (1 + \|\mathbf{n}\|^2)^m.$$

Poisson Equation with Dirichlet conditions

We note the Laplacian of a function u by $\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy}$.
Let be $\Omega = [0, 2\pi]^2$ and $\partial\Omega$ its border.

Theorem 2: *Equation de Poisson avec condition de Dirichlet*

- Let $f \in L^2([0, 2\pi]^2)$ (the *heat source*).
- Consider the Poisson equation in a square in isothermal conditions (temperature in the square border is fixed to 0). This is called *Poisson equation with Dirichlet condition*.
- The equation that represents the temperature inside the square can be written as

$$\begin{cases} -\Delta u = f, & \text{with } f \in L^2([0, 2\pi]^2) \quad (*) \\ u = 0 & \text{in } \partial\Omega. \end{cases}$$

Then, there exists a unique $u \in L^2([0, 2\pi]^2)$ solution (in the *distribution sense*). This solution is an odd (impaire) function in $H^2_{\text{per}}([-2\pi, 2\pi]^2)$ (a 4π- periodic function).

Proof In order to naturally have the boundary condition $u = 0$ it is convenient to use the sine basis decomposition

$$\left(\sin\left(\tfrac{kx}{2}\right)\sin\left(\tfrac{ly}{2}\right)\right)_{k,l}, \text{ with } k,l \in \mathbb{N}^*$$

Hence, we write

$$u(x,y) = \textstyle\sum_{k,l\in\mathbb{N}^*} c_{k,l}(u)\sin\left(\tfrac{kx}{2}\right)\sin\left(\tfrac{ly}{2}\right)$$

$$f(x,y) = \textstyle\sum_{k,l\in\mathbb{N}^*} c_{k,l}(f)\sin\left(\tfrac{kx}{2}\right)\sin\left(\tfrac{ly}{2}\right)$$

The equation $(*)$ can be written

$$\sum_{k,l\in\mathbb{N}^*} c_{k,l}(f)\sin\left(\frac{kx}{2}\right)\sin\left(\frac{ly}{2}\right) = \sum_{k,l\in\mathbb{N}^*} c_{k,l}(u)\left(\frac{k^2+l^2}{4}\right)\sin\left(\frac{kx}{2}\right)\sin\left(\frac{ly}{2}\right)$$

By the unicity of the Fourier decomposition, we have

$$c_{k,l}(f) = c_{k,l}(u)\tfrac{k^2+l^2}{4}, \quad \forall k,l \in \mathbb{N}^*$$

Then

$$u(x,y) = 4\sum_{k,l\in\mathbb{N}^*}\frac{c_{k,l}(f)}{k^2+l^2}\sin\left(\frac{kx}{2}\right)\sin\left(\frac{ly}{2}\right)$$

Notice that, $u \in H^2_{\text{per}}([-2\pi,2\pi]^2)$, $u \notin C^2$, but $u \in C^0$. Since u is odd, then we have $u(x,y) = 0$ in $\partial\Omega$. □

Poisson Equation with Neumann conditions

Let be $\Omega = [0, 2\pi]^2$ and $\partial\Omega$ its border.

Theorem 3: *Equation de Poisson avec condition de Neumann*

- Let $f \in L^2(\Omega)$, such that $\int_\Omega f(x,y)dxdy = 0$ (the *heat source*).
- Consider the Poisson equation in a square with adiabatic boundary (e.g. heat flux through the boundary is zero). This is called *Poisson equation with Neumann condition*.
- The equation that represents the temperature inside the square can be written as

$$\begin{cases} -\Delta u = f, & \text{with } f \in L^2(\Omega),\ \int_\Omega f = 0 \quad (*) \\ \frac{\partial u}{\partial n} = 0 & \text{in } \partial\Omega. \end{cases}$$

Then, there exists $u \in L^2([0, 2\pi]^2)$ solution (in the *distribution sense*).

The solution satisfies $\frac{\partial u}{\partial n} = 0$ in the sense that is an even (paire) function.

This solution is unique up to a constant and belongs to $H^2_{\text{per}}([-2\pi, 2\pi]^2)$ (a 4π- periodic function).

Proof In order to naturally have the boundary condition $\frac{\partial u}{\partial n} = 0$ it is convenient to use the cosine basis decomposition

$$\left(\cos\left(\frac{kx}{2}\right)\cos\left(\frac{ly}{2}\right)\right), \text{ with } k, l \in \mathbb{N}.$$

Hence, we write

$$u(x,y) = \sum_{k,l\in\mathbb{N}} c_{k,l}(u)\cos\left(\frac{kx}{2}\right)\cos\left(\frac{ly}{2}\right)$$

$$f(x,y) = \sum_{k,l\in\mathbb{N}} c_{k,l}(f)\cos\left(\frac{kx}{2}\right)\cos\left(\frac{ly}{2}\right)$$

with

$$c_{k,l} = c(k)c(l)\int_\Omega u(t,s)\cos\left(\frac{kt}{2}\right)\cos\left(\frac{ls}{2}\right)dtds$$

where $c(n) = \frac{1}{\pi}$ iff $n = 0$ or $c(n) = \frac{1}{2\pi}$ otherwise.

The equation $(*)$ can be written

$$\sum_{k,l\in\mathbb{N}} c_{k,l}(f)\cos\left(\frac{kx}{2}\right)\cos\left(\frac{ly}{2}\right) = \sum_{k,l\in\mathbb{N}} c_{k,l}(u)\left(\frac{k^2+l^2}{4}\right)\cos\left(\frac{kx}{2}\right)\cos\left(\frac{ly}{2}\right).$$

By the unicity of the Fourier decomposition, we have

$$c_{k,l}(u)\frac{k^2+l^2}{4} = c_{k,l}(f) \quad \forall k,l\in\mathbb{N}.$$

Then,

$$c_{k,l}(u) = 4\frac{c_{k,l}(f)}{k^2+l^2},\ (k,l)\neq(0,0)$$

Recall that $\int f = 0$ and hence $c_{0,0}(f) = 0$.

Proof (cont.)

Next, we have

$$u(x,y) = c + 4 \sum_{k,l \in \mathbb{N}^*} \frac{c_{k,l}(f)}{k^2 + l^2} \cos\left(\frac{kx}{2}\right) \cos\left(\frac{ly}{2}\right)$$

where the constant c is not determined by the system so there is one degree of freedom ($c_{0,0}(u)$ not defined).

- Notice that, $u \in H^2_{\text{per}}([-2\pi, 2\pi]^2)$.
- Remark that $u \notin C^2$, but $u \in C^0$ (the equation only makes sense in the distribution sense).
- Since u is even (paire), we have $\frac{\partial u}{\partial n}(x,y) = 0$ in $\partial\Omega$. (note that the function may not be differentiable in the boundary, however if it is, the derivative must be zero).

$\square$

Coercivity

We say that a symmetric bilinear form is coercive if there exists a constant $C > 0$ such that

$$a(u,u) \geq c\|u\|^2 \quad \forall u \in H$$

Remark

A lot of problems in physics, mechanics, signal processing can be reduced to minimize in a Hilbert space (H), an energy of the form

$$(*) \quad E(u) = \frac{1}{2}a(u,u) - (b,u)$$

where $b \in H$ and $a(.,.)$ is bilinear, continuous and coercive.

Lax-Milgram Lemma

Theorem 5: Let H be a Hilbert separable space and E an energy functional of the type

$$(*) \quad E(u) = \frac{1}{2} a(u,u) - (b,u),$$

with $a(\cdot,\cdot)$ **bilinear**, **symmetric**[1], **continuous** and **coercive**.

Then, there exists a unique u, minimizer of $E(u)$ in H.

Moreover, u is characterized by the following variational equation

$$\forall v \in H, \quad \mathrm{Re}(a(u,v)) = (b,v).$$

[1] $a(u,v) = \overline{a(u,v)}$

End of last lecture summary.

The Variational Interpretation

Proposition 1: Let $\Omega = [0, 2\pi]^2$ and $f \in L^2_{\text{per}}(\mathbb{R}^2)$. Then there exists a unique function $u \in H^1_{\text{per}}(\mathbb{R}^2)$ minimizer of the energy

$$E(u) = \frac{1}{2}\int_\Omega |Du|^2(x,y)dxdy - \int_\Omega f(x,y)u(x,y)$$

and satisfying $u = 0$ in $\partial\Omega$.

This solution is the restriction to Ω of an odd function belonging to $H^2_{\text{per}}([-2\pi, 2\pi]^2)$ (4π-periodic functions).

This solution is the same as the one from the Poisson equation with Dirichlet conditions:

$$-\Delta u = f, \quad u = 0 \ \text{ in } \ \partial\Omega$$

Proof In order to naturally have the boundary condition $u = 0$ it is convenient to use the sine basis decomposition of $L^2([0, 2\pi]^2)$

$$\left(\sin\left(\tfrac{kx}{2}\right)\sin\left(\tfrac{ly}{2}\right)\right), \text{ with } k, l \in \mathbb{N}^*$$

Hence, we write

$$u(x, y) = \textstyle\sum_{k,l\in\mathbb{N}^*} c_{k,l}(u) \sin\left(\tfrac{kx}{2}\right) \sin\left(\tfrac{ly}{2}\right)$$

$$f(x, y) = \textstyle\sum_{k,l\in\mathbb{N}^*} c_{k,l}(f) \sin\left(\tfrac{kx}{2}\right) \sin\left(\tfrac{ly}{2}\right)$$

By using the respective Fourier expansions, the energy can be re-written as

$$\begin{aligned}\frac{1}{2}\int_\Omega |Du|^2(x, y)dxdy &= \frac{1}{2}\int_\Omega \Big(\sum_{k,l\in\mathbb{N}^*} c_{k,l}(u)\frac{k}{2}\cos\frac{kx}{2}\sin\frac{ly}{2}\Big)^2 + \\ &\qquad \frac{1}{2}\int_\Omega \Big(\sum_{k,l\in\mathbb{N}^*} c_{k,l}(u)\frac{l}{2}\sin\frac{kx}{2}\cos\frac{ly}{2}\Big)^2 \\ &= \sum_{k,l\in\mathbb{N}^*} \frac{k^2 + l^2}{8} c_{k,l}(u)^2\end{aligned}$$

$$\begin{aligned}\int_\Omega f(x, y)u(x, y) &= \int_\Omega \Big(\sum_{k,l\in\mathbb{N}^*} c_{k,l}(f)\sin\frac{kx}{2}\sin\frac{ly}{2}\Big)\Big(\sum_{k,l\in\mathbb{N}^*} c_{k,l}(u)\sin\frac{kx}{2}\sin\frac{ly}{2}\Big) \\ &= \sum_{k,l\in\mathbb{N}^*} c_{k,l}(u)c_{k,l}(f)\end{aligned}$$

Proof (cont. 1) Then, regrouping the energy $E(u)$:

$$E(u) = \sum_{k,l \in \mathbb{N}^*} \frac{k^2 + l^2}{8} c_{k,l}(u)^2 - c_{k,l}(u) c_{k,l}(f)$$

We need to minimize a sum with infinite terms. However, each term depends on a different variable: $c_{k,l}(u)$.

Thus, we can minimize each term separately (differentiate and $= 0$) and we get

$$c_{k,l}(u) = \frac{4}{k^2 + l^2} c_{k,l}(f) \quad \forall k, l \in \mathbb{N}^*$$

Notice that is the solution of the Poisson equation with the Dirichlet condition!

$$u(x, y) = 4 \sum_{k,l \in \mathbb{N}^*} \frac{c_{k,l}(f)}{k^2 + l^2} \sin\left(\frac{kx}{2}\right) \sin\left(\frac{ly}{2}\right)$$

Finally $u \in H^2_{\text{per}}([-2\pi, 2\pi]^2)$, $u \notin C^2$, but $u \in C^0$. Since u is odd, then we have $u(x, y) = 0$ in $\partial\Omega$. □

Poisson Editor

Proposition 2: Let $\Omega = [0, 2\pi]^2$ and $V = (v_1, v_2) \in (L^2_{\text{per}}(\mathbb{R}^2))^2$ a vector field. Then there exists a function $u \in H^1_{\text{per}}(\mathbb{R}^2)$ minimizer of the energy

$$E(u) = \frac{1}{2}\int_\Omega |Du - V|^2(x,y)dxdy = \int_\Omega \left((u_x - v_1)^2 + (u_y - v_2)^2\right) dxdy$$

and satisfying $\frac{\partial u}{\partial n} = 0$ in $\partial\Omega$.

This solution is the restriction to Ω of an even (paire) function belonging to $H^1_{\text{per}}([-2\pi, 2\pi]^2)$ (4π-periodic functions).

This solution is the same as the one from the Poisson equation with Neumann conditions:

$$-\Delta u = -\text{div}(V) = -\frac{\partial v_1}{\partial x} - \frac{\partial v_2}{\partial y}, \quad \frac{\partial u}{\partial n} = 0 \text{ in } \partial\Omega$$

Proof In order to naturally have the boundary condition $\frac{\partial u}{\partial n} = 0$ it is convenient to use the cosine basis decomposition for elements in $L^2([0, 2\pi]^2)$

$$\left(\cos\left(\tfrac{kx}{2}\right)\cos\left(\tfrac{ly}{2}\right)\right), \text{ with } k, l \in \mathbb{N}$$

Hence, we write

$$u(x, y) = \textstyle\sum_{k,l\in\mathbb{N}} c_{k,l}(u) \cos\left(\tfrac{kx}{2}\right) \cos\left(\tfrac{ly}{2}\right)$$

$$v_1(x, y) = \textstyle\sum_{k,l\in\mathbb{N}} c_{k,l}(v_1) \cos\left(\tfrac{kx}{2}\right) \cos\left(\tfrac{ly}{2}\right)$$

$$v_2(x, y) = \textstyle\sum_{k,l\in\mathbb{N}} c_{k,l}(v_2) \cos\left(\tfrac{kx}{2}\right) \cos\left(\tfrac{ly}{2}\right)$$

By using the respective Fourier expansions for u_x, u_y, (derivatives in the distribution sense) and v_1, v_2 the Poisson energy can be re-written as

$$E(u) = \frac{1}{2} \sum_{k,l\in\mathbb{N}} \left(\frac{k}{2} c_{k,l}(u) - c_{k,l}(v_1)\right)^2 + \left(\frac{l}{2} c_{k,l}(u) - c_{k,l}(v_2)\right)^2$$

Proof (cont. 1) Then, regrouping the energy $E(u)$:

$$E(u) = \frac{1}{2} \sum_{k,l \in \mathbb{N}^*} \frac{k^2+l^2}{4} c_{k,l}(u)^2 - \Big(k c_{k,l}(v_1) + l c_{k,l}(v_2) \Big) c_{k,l}(u) + c_{k,l}(v_1)^2 + c_{k,l}(v_2)^2$$

Each term depends on a different $c_{k,l}(u)$. We need to minimize

$$\left(\frac{k^2+l^2}{4} \right) c_{k,l}(u)^2 - \Big(k c_{k,l}(v_1) + l c_{k,l}(v_2) \Big) c_{k,l}(u)$$

so if we differentiate and $= 0$, we get

$$c_{k,l}(u) = 2 \frac{k c_{k,l}(v_1) + l c_{k,l}(v_2)}{k^2+l^2}, \quad \forall k, l > 0$$

- u is in $H^1_{\text{per}}([-2\pi, 2\pi]^2)$ since $\frac{k}{2} c_{k,l}(u) \in l^2(\mathbb{Z}^2)$ (so $u_x \in L^2_{\text{per}}$).
- The solution is unique up to a constant (the term $c_{0,0}(u)$ is not set). If u is solution then $u + C$ for C constant is also solution. Thus,

$$u(x,y) = C + \sum_{k,l \in \mathbb{N}^*} 2 \frac{k c_{k,l}(v_1) + l c_{k,l}(v_2)}{k^2+l^2} \cos\left(\frac{kx}{2} \right) \cos\left(\frac{ly}{2} \right)$$

- Same solution as the Poisson equation with the Neumann condition for

$$f = -\text{div}(V) = -(v_1)_x - (v_2)_y$$

Proposition 3: Let u be the solution of the Poisson editor for a gradient field $V = (v_1, v_2)$. Then u satisfies the equation

$$\Delta u = \mathrm{div}(V)$$

in the distribution sense.

Proof Consider a perturbation $t\varphi$, with $\varphi \in C_c^\infty(\Omega)$ and $t \in \mathbb{R}$. If u is a minimum then

$$E(u + t\varphi) = \int_\Omega (D(u + t\varphi) - V)^2 \geq \int_\Omega (Du - V)^2 = E(u)$$

Then,

$$\int_\Omega (Du - V)^2 + 2tD\varphi \cdot (Du - V) + t^2 Du \cdot Du \geq \int_\Omega (Du - V)^2$$

Since u is a minimum, the derivative in $t = 0$ should be zero

$$\int_\Omega D\varphi \cdot (Du - V) = 0$$

This is the same as

$$\int_\Omega \varphi_x(u_x - v_1) + \varphi_y(u_y - v_2) = 0$$

Since u_x, u_y, v_1 and v_2 are in L^2_{loc} they are also distributions:

$$< u_x - v_1, \varphi_x > + < u_y - v_2, \varphi_y >= 0$$

Next, from the definition of distribution derivative

$$- < u_{xx} - (v_1)_x, \varphi > - < u_{yy} - (v_2)_y, \varphi >= - < u_{xx} + u_{yy} - (v_1)_x - (v_2)_y, \varphi >= 0$$

and thus

$$\Delta u - \operatorname{div}(V) = 0.$$

$\square$

Periodic + Smooth decomposition

Proposition 4: Let u be a function such that $\partial^{\mathbf{i}} u \in L^2([0,2\pi]^2)$ for $|\mathbf{i}| \leq 2$. We periodize Δu.

Then, there exists a function $v \in H^2_{\text{per}}([0,2\pi]^2)$ such that

$$\Delta v = \Delta u \quad \text{in } [0,2\pi]^2$$

This function is unique up to an additive constant. The difference $w = v - u$ verifies $\Delta w = 0$ and hence it is smooth in $[0,2\pi]^2$.

Proof

Consider the standard Fourier basis in $[0, 2\pi]^2$. We can expand v and Δu in this basis

$$v = \sum_{(k,l)\in\mathbb{Z}^2} c_{k,l}(v)\mathrm{e}^{ikx}\mathrm{e}^{ily}$$

$$\Delta u = \sum_{(k,l)\in\mathbb{Z}^2} c_{k,l}(\Delta u)\mathrm{e}^{ikx}\mathrm{e}^{ily} = \Delta v$$

We can differentiate term-a-term v in the distribution sense,

$$\Delta v = -\sum_{(k,l)\in\mathbb{Z}^2} (k^2 + l^2)c_{k,l}(v)\mathrm{e}^{ikx}\mathrm{e}^{ily} = \sum_{k,l\in\mathbb{Z}} c_{k,l}(\Delta u)\mathrm{e}^{ikx}\mathrm{e}^{ily}$$

then from the unicity of the Fourier decomposition for distributions

$$-(k^2 + l^2)c_{k,l}(v) = c_{k,l}(\Delta u) \iff c_{k,l}(v) = -\frac{c_{k,l}(\Delta u)}{k^2 + l^2} \quad k, l \neq (0, 0)$$

Thus,

$$v = -\sum_{(k,l)\in\mathbb{Z}^2 \ \{0,0\}} \frac{c_{k,l}(\Delta u)}{k^2 + l^2}\mathrm{e}^{ikx}\mathrm{e}^{ily}$$

This determines the solution up to an additive constant ($c_{0,0}(v)$ is not fixed). $\square$

Periodic + Smooth decomposition

The P+S decomposition permits to visualize the Fourier spectrum getting rid of the boundary effects (e.g. discontinuities when periodizing). The periodic decomposition inherits all the image details since its laplacian is the same.

original Lena image

its periodic component p

L. Moisan, *Periodic plus Smooth Image Decomposition*, Journal of Math. Imag. and Vision, 2011.

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(replicated)

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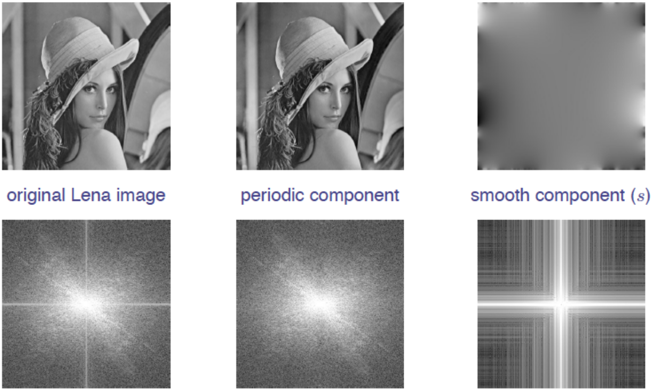
Periodic + Smooth decomposition

The P+S decomposition permits to visualize the Fourier spectrum getting rid of the boundary effects (e.g. discontinuities when periodizing). The periodic decomposition inherits all the image details since its laplacian is the same.

$$u = p + s$$



original Lena image | periodic component | smooth component (s)

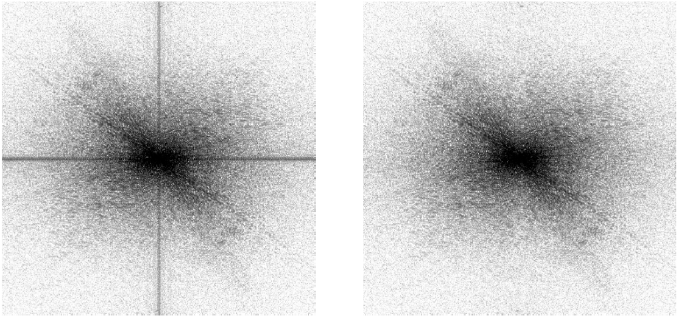
corresponding Discrete Fourier Transforms (log-modulus)

L. Moisan, *Periodic plus Smooth Image Decomposition*, Journal of Math. Imag. and Vision, 2011.

Periodic + Smooth decomposition

The P+S decomposition permits to visualize the Fourier spectrum getting rid of the boundary effects (e.g. discontinuities when periodizing). The periodic decomposition inherits all the image details since its laplacian is the same.

Spectrum (log-modulus)



original Lena image   periodic component

L. Moisan, *Periodic plus Smooth Image Decomposition*, Journal of Math. Imag. and Vision, 2011.

Unicity of Fourier coefficients

Proposition 5: There exists $\chi \in C_0^\infty(\mathbb{R}^2)$ such that

$$\sum_{\mathbf{k}\in\mathbb{Z}^2} \chi(\mathbf{x} + 2\mathbf{k}\pi) = 1$$

We call χ a periodic partition of the unity.

Proof

Consider a function $\varphi \in C_0^\infty$ such that

$$\varphi \geq 0, \quad \text{in } \mathbb{R}^2$$
$$\varphi > 0, \quad \text{in } [0, 2\pi]^2$$

Then, we set

$$\chi(\mathbf{x}) = \frac{\varphi(\mathbf{x})}{\sum_{\mathbf{k}\in\mathbb{Z}^2} \varphi(\mathbf{x} + 2\mathbf{k}\pi)}$$

Then

$$\sum_{\mathbf{k}\in\mathbb{Z}^2} \chi(\mathbf{x} + 2\mathbf{k}\pi) = 1$$

Since φ has compact support the sum in the denominator is finite for all $\mathbf{x}$. $\square$

Lemma 1: Let $\varphi \in C_0^\infty(\mathbb{R}^2)$ be a test function and $\tilde{\varphi}(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^2} \varphi(\mathbf{x} + 2\pi\mathbf{k})$ its periodization.

Then, the Fourier expansion of $\tilde{\varphi}$ can be written as

$$\tilde{\varphi}(\mathbf{x}) = \sum_{\mathbf{m}\in\mathbb{Z}^2} c_m(\tilde{\varphi})\mathrm{e}^{i\mathbf{m}\cdot\mathbf{x}}$$

with

$$c_{\mathbf{m}}(\tilde{\varphi}) = \frac{1}{2\pi^2}\int_{\mathbb{R}^2} \varphi(\mathbf{x})\mathrm{e}^{-i\mathbf{m}\cdot\mathbf{x}} d\mathbf{x}.$$

Proof

The Fourier coefficients of $\tilde{\varphi}$ are

$$\begin{aligned}
c_{\mathbf{m}}(\tilde{\varphi}) &= \frac{1}{2\pi^2}\int_{[0,2\pi]^2} \tilde{\varphi}(\mathbf{x})\mathrm{e}^{-i\mathbf{m}\cdot\mathbf{x}}d\mathbf{x} \\
&= \frac{1}{2\pi^2}\int_{[0,2\pi]^2} \sum_{\mathbf{k}\in\mathbb{Z}^2} \varphi(\mathbf{x}+2\pi\mathbf{k})\mathrm{e}^{-i\mathbf{m}\cdot\mathbf{x}}d\mathbf{x} \\
&= \frac{1}{2\pi^2}\int_{[0,2\pi]^2} \sum_{\mathbf{k}\in\mathbb{Z}^2} \varphi(\mathbf{x}+2\pi\mathbf{k})\mathrm{e}^{-i\mathbf{m}\cdot(\mathbf{x}+2\pi\mathbf{k})}d\mathbf{x} \\
&= \frac{1}{2\pi^2}\sum_{\mathbf{k}\in\mathbb{Z}^2} \int_{[0,2\pi]^2+2\pi\mathbf{k}} \varphi(\mathbf{x})\mathrm{e}^{-i\mathbf{m}\cdot\mathbf{x}}d\mathbf{x} \\
&= \frac{1}{2\pi^2}\int_{\mathbb{R}^2} \varphi(\mathbf{x})\mathrm{e}^{-i\mathbf{m}\cdot\mathbf{x}}d\mathbf{x}
\end{aligned}$$

$\square$

Corollary 1: For all $\mathbf{m}_0 \in \mathbb{Z}^2$, there exists a function $\varphi \in C_0^\infty$ such that

$$\forall \mathbf{m} \in \mathbb{Z}^2, \quad \int_{\mathbb{R}^2} \varphi(\mathbf{x}) \mathrm{e}^{i\mathbf{m}\cdot\mathbf{x}} d\mathbf{x} = \delta_{\mathbf{m},\mathbf{m}_0}.$$

$\delta_{\mathbf{m},\mathbf{m}_0} = 1$ if $\mathbf{m} = \mathbf{m}_0$; 0 otherwise. (Kronecker delta)

Proof
Take

$$\varphi(\mathbf{x}) = \chi(\mathbf{x})\mathrm{e}^{-\mathbf{m}_0\cdot\mathbf{x}},$$

where $\chi(\mathbf{x})$ is a periodic partition of the unity. Then,

$$\begin{aligned}\tilde{\varphi}(\mathbf{x}) &= \sum_{\mathbf{k}\in\mathbb{Z}^2} \varphi(\mathbf{x}+2\mathbf{k}\pi)\\ &= \sum_{\mathbf{k}\in\mathbb{Z}^2} \chi(\mathbf{x}+2\mathbf{k}\pi)\mathrm{e}^{-\mathbf{m}_0\cdot(\mathbf{x}+2\mathbf{k}\pi)}\\ &= \sum_{\mathbf{k}\in\mathbb{Z}^2} \chi(\mathbf{x}+2\mathbf{k}\pi)\mathrm{e}^{-\mathbf{m}_0\cdot\mathbf{x}}\\ &= \mathrm{e}^{-\mathbf{m}_0\cdot\mathbf{x}.}.\end{aligned}$$

We apply Lemma 1 to φ, and from the unicity of the Fourier coefficients (for a function)

$$\tilde{\varphi}(\mathbf{x}) = \mathrm{e}^{-\mathbf{m}_0\cdot\mathbf{x}} = \sum_{\mathbf{m}\in\mathbb{Z}^2} c_m(\tilde{\varphi})\mathrm{e}^{i\mathbf{m}\cdot\mathbf{x}}$$

Then the only Fourier coefficient different from zero is $c_{-\mathbf{m}_0}(\tilde{\varphi}) = 1$. Finally,

$$c_{-\mathbf{m}}(\tilde{\varphi}) = \int_{\mathbb{R}^2} \varphi(\mathbf{x})\mathrm{e}^{i\mathbf{m}\cdot\mathbf{x}} d\mathbf{x} = 1 \quad \text{if } \mathbf{m} = \mathbf{m}_0; \quad 0 \text{ otherwise.}$$

$\square$

Theorem : *Unicity of the Fourier coefficients*
If u is a distribution with the Fourier expansion

$$u = \sum_{\mathbf{m}\in\mathbb{Z}^2} c_{\mathbf{m}} \mathrm{e}^{i\mathbf{m}\cdot\mathbf{x}} = 0$$

then $\forall \mathbf{m}$, $c_{\mathbf{m}} = 0$.

Proof

We know that $\forall \varphi \in C_0^\infty$,

$$0 =< \sum_{\mathbf{m}\in\mathbb{Z}^2} c_{\mathbf{m}} \mathrm{e}^{i\mathbf{m}\cdot\mathbf{x}}, \varphi > = \sum_{\mathbf{m}\in\mathbb{Z}^2} c_{\mathbf{m}} < \mathrm{e}^{i\mathbf{m}\cdot\mathbf{x}}, \varphi > = \sum_{\mathbf{m}\in\mathbb{Z}^2} c_{\mathbf{m}} \int_{\mathbb{R}^2} \varphi(\mathbf{x}) \mathrm{e}^{i\mathbf{m}\cdot\mathbf{x}} d\mathbf{x}.$$

Since the last statement is true $\forall \varphi \in C_0^\infty$, then in particular we can choose φ as shown by the last Corollary. Conclude: $c_{\mathbf{m}} = 0, \forall \mathbf{m}$. □

What do the functions in Sobolev spaces look like?

Fourier coefficients decrease at varying speed:

$$|c_k(f)| = (|k| + 1)^{-\alpha} \qquad \arg(c_k(f)) \sim \mathcal{U}[0, 2\pi]$$

Then, we apply the inverse Fourier transform and we get:

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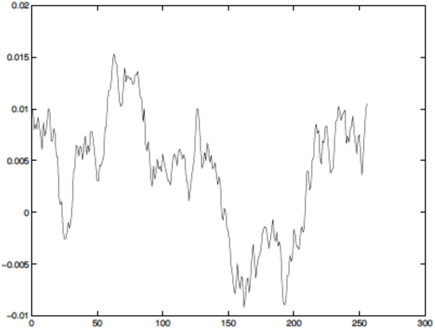
$\alpha = 1.01$



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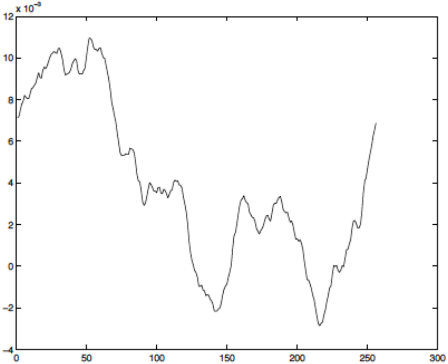
Then, we apply the inverse Fourier transform and we get:

$\alpha = 1.5$



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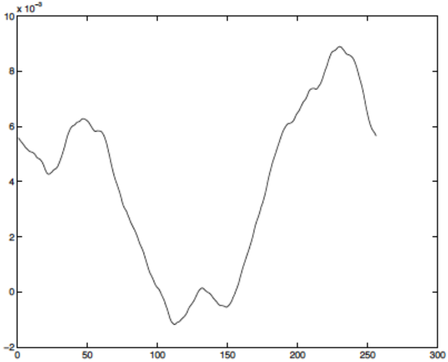
$$|c_k(f)| = (|k| + 1)^{-\alpha} \qquad \arg(c_k(f)) \sim \mathcal{U}[0, 2\pi]$$

Then, we apply the inverse Fourier transform and we get:

$\alpha = 2$



What do the functions in Sobolev spaces look like?

Fourier coefficients decrease at varying speed:

$$|c_{i,j}(f)| = (i^2 + j^2 + 1)^{-\frac{\alpha}{2}} \qquad \arg(c_{i,j}(f)) \sim \mathcal{U}[0, 2\pi]$$

Then, we apply the inverse Fourier transform and we get:

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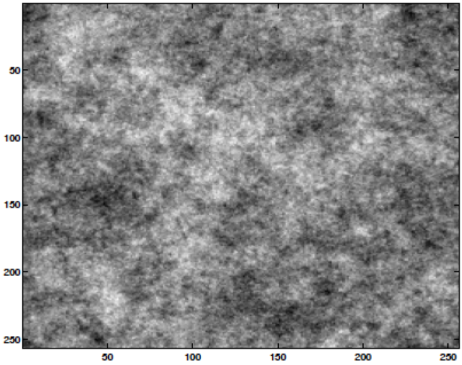

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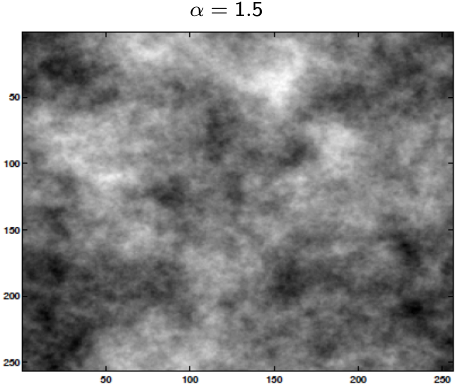

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