SOME COMMENTS ON MANIPULATING SIGNALS IN FOURIER DOMAIN

Symptoms: If your image translation only works for integer displacements, then you'd better read this.

About the trigonometric polynomials. The definition of the DFT presented in the course (and "polycopié") differs from the one computed by MATLAB. In particular we define DFT coefficients as

$$\widetilde{u}_n = \frac{1}{N}\sum_{k=0}^{N-1} u_k \exp\left(-\frac{2i\pi kn}{N}\right) \quad \text{for} \quad \mathbf{n} = -\frac{\mathbf{N}}{\mathbf{2}}, ..., \frac{\mathbf{N}}{\mathbf{2}} - \mathbf{1}, \tag{1}$$

while MATLAB and other software based on FFTW compute

$$a[n] = \sum_{k=1}^{N} x[k] \exp\left(-\frac{2\pi i(k-1)(n-1)}{N}\right), \quad \text{for} \quad \mathbf{n} = \mathbf{1}, ..., \mathbf{N}. \tag{2}$$

The key observation is that, while in (1) the frequency of the exponentials goes from $-\frac{N}{2}, ..., \frac{N}{2} - 1$, in (2) it goes from $0, ...N-1$.

Since the exponentials of the form $\exp(2\pi ik/N)$ are N-periodic it is possible to deduce a bijection between $(a[n])_{n=1...N}$ and $(\widetilde{u}_n)_{n=-\frac{N}{2},...,\frac{N}{2}-1}$.

Interpolation of a signal by trigonometric polynomials. Although the coefficients in (1) and (2) are strictly the same (up to a factor $1/N$ and its order), to interpolate the samples we must use the following polynomial

$$u(x) = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} \widetilde{u}_n \exp\left(\frac{2i\pi xn}{N}\right),$$

where the coefficients (and the respective exponential frequencies) go from $-\frac{N}{2}, ..., \frac{N}{2} - 1$ (as in Proposition 6.1).

Therefore, for manipulating a signal in the Fourier domain we must remit to this polynomial. For example consider the translation of a 1D signal presented in Proposition 6.9. In order to translate the signal by a shift α we need to multiply its Fourier coefficients by $\exp\left(-\frac{2\pi in\alpha}{N}\right)$, **where $\mathbf{n} \in [-\frac{\mathbf{N}}{\mathbf{2}}, ...\frac{\mathbf{N}}{\mathbf{2}} - \mathbf{1}]$.**

About the `fftshift` command. The `fftshift` command can be used to re-organize the Fourier coefficient vector computed by MATLAB in a way consistent with our formulation (see Figure 1). However since it is only a re-organization it is not strictly necessary to use it and by properly manipulating the positions in the vectors the same result can be achieved.

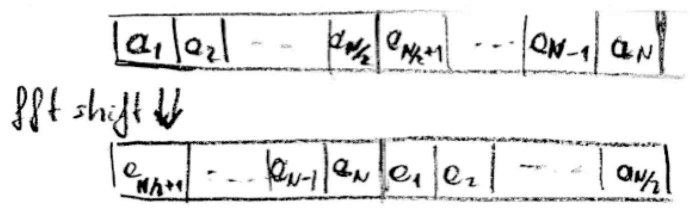

FIGURE 1. fftshift diagram