(see figure)

We deduce the others λ

$$\begin{cases} -4\lambda_0 = -2 + 4\cos^2(\theta)\sin^2(\theta) \\ \lambda_1 = \cos^2(\theta)(\cos^2(\theta) - sin^2(\theta)) \\ \lambda_2 = \sin^2(\theta)(\sin^2(\theta) - cos^2(\theta)) \\ \lambda_3 = \cos^2(\theta)\sin^2(\theta) + 0.5\sin(\theta)\cos(\theta) \\ \lambda_4 = \cos^2(\theta)\sin^2(\theta) - 0.5\sin(\theta)\cos(\theta) \end{cases}$$

We still consider the stencil 3*3. And, we want now to have an estimation of the second derivative of u in the direction ξ, exact for a set of polynomials. Of course, we want the degree of polynomials as high as possible.

In order to have `[Consistency]` , we want at least the polynomials of degree two. that gives us six independant relations on our α_{ij}.

We remark that on our stencil $x^3 = x$ and $y^3 = y$ and that $(x)_{\xi\xi} = (x^3)_{\xi\xi} = (y)_{\xi\xi} = (y^3)_{\xi\xi} = 0$. Then, our calcul will be exact for the two more polynomials : x^3 and y^3.

invariance by rotation of radius $\frac{\pi}{4}$,

$$(ii) \qquad \lambda_0(\theta + \frac{\pi}{2} = \lambda_0(\theta)$$

symetry with respect to the axe i+j,

$$(iii) \qquad \lambda_0(\frac{\pi}{2} - \theta) = \lambda_0\theta$$

Of course, we seek a smooth function for $\lambda_0(\theta)$. We shall take it, C^1. Then,

$$(i) \Rightarrow \lambda_0'(0) = 0$$
$$(iii) \Rightarrow \lambda_0'(\frac{\pi}{4}) = 0$$

(i) and (ii) imply that λ_0 is $\frac{\pi}{2}$ periodic, and symetric with respect to $\frac{\pi}{4}$. We shall now study it on $[0, \frac{\pi}{4}]$.

Let us sumerize We are looking for a function λ_0 on the interval $[0, \frac{\pi}{4}]$, such that :

$$(11) \qquad \begin{cases} \lambda_0(0) = 0.5 \\ \lambda_0(\frac{\pi}{4} = 0.25 \\ \lambda_0'(0) = 0 \\ \lambda_0'(\frac{\pi}{4}) = 0 \\ \lambda_1\lambda_2\lambda_3\lambda_4 \text{as “positive” as possible} \end{cases}$$

For θ in $[0, \frac{\pi}{4}]$,

$$\lambda_1 \geq \lambda_2 \qquad \text{and} \qquad \lambda_3 \geq \lambda_4$$

But

$$\lambda_2(\theta) = 0 \Rightarrow \lambda_0(\theta) \geq \frac{\cos^2(\theta)}{2}$$
$$\lambda_4(\theta) = 0 \Rightarrow \lambda_0(\theta) \leq \frac{1 - \sin(\theta)\cos(\theta)}{2}$$

But

$$\frac{\cos^2(\theta)}{2} \geq \frac{1 - \sin(\theta)\cos(\theta)}{2}$$

Then, if we choice $\lambda_0(\theta) \geq \frac{\cos^2(\theta)}{2}$ we have $\lambda_4(\theta)$ very negative. And, now if we take $\lambda_0(\theta) \leq \frac{1-\sin(\theta)\cos(\theta)}{2}$ we obtain $\lambda_2(\theta)$ very negative.

We prefer to choice λ_0 between these two functions, and then to have λ_2 and $lambda_4$ negative, but weekly.

The periodicity of λ_0 invite us to choice it as a trigonometric polynomial, of degree as small as possible for numerical simplicity. The simplest polynomial verifing (??), and between the two functions is

$$0.5 - \cos^2(\theta) + \cos^4(\theta)$$

Then, we can write

$$u_{\xi\xi} = \sin^2(\theta)u_{xx} - 2\sin(\theta)\cos(\theta)u_{xy} + \cos^2(\theta)u_yy \tag{9}$$

We are looking to write $u_{\xi\xi}$ as a linear combination of the value of u in the fixed stencil 3*3. Of course, the coefficients of the linear combination may depend on ξ.

Because the direction of the gradient (and then ξ) is defined modulo pi, by symetry we can assume that the coefficients of symetrical points with respect to the central point are the same.

In order to have consistancy, we must find $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4$, such that :

$$u_{\xi\xi} = \frac{1}{\Delta x^2}(-4\lambda_0 u_{i,j}+\lambda_1(u_{i+1,j}+u_{i-1,j})+\lambda_2(u_{i,j+1}+u_{i,j-1})+\lambda_3(u_{i-1,j-1}+u_{i+1,j+1})+\lambda_4(u_{i-1,j+1}+u_{i+1,j} \tag{10}$$

We write

$$u_{i+1,j} = u_{i,j} + h(u_x)_{i,j} + \frac{h^2}{2}(u_x x)_{i,j} + o(\Delta x^2)$$

And the same thing for the others points. By subtituated these relations in (10) and by using relation (1), we obtain four relations between our five coefficients

$$\begin{cases} \lambda_1(\theta) = 2\lambda_0(\theta) - \sin^2\theta \\ \lambda_2(\theta) = 2\lambda_0(\theta) - \cos^2\theta \\ \lambda_3(\theta) = -\lambda_0(\theta) + 0.5(\sin\theta\cos\theta + 1) \\ \lambda_4(\theta) = -\lambda_0(\theta) - 0.5(\sin\theta\cos\theta + 1) \end{cases}$$

It remains only one degree of liberty for our coefficients : for exemple the choice of λ_0.

Now, in order to have the [L^∞ stability] , we must have the following relation

$$4\lambda_0 \geq 2(|\lambda_1| + |\lambda_2| + |\lambda_3| + |\lambda_4|)$$

But $2\lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 - 4\lambda_0 = 0$, then $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ must be nonnegative.

But because of the relations between our coefficients, it is impossible to assume such thing, but for particular values of $\theta(0, \frac{\pi}{4}, \frac{\pi}{2}, ...)$.

Then, for the choice of λ_0, we impose the [L^∞ stability] , for the particulars values of θ, when we can have it. We have

$$\lambda_0(0) = 0.5$$

$$\lambda_0(\frac{\pi}{4} = 0.25$$

Then, we write some obvious geometric invariances

symetry with respect to the central point,

$$(i) \qquad \lambda_0(\theta + \pi) = \lambda_0(\theta)$$

$\lambda_0(\theta + \frac{\pi}{2}) = \lambda_0(\theta)$ $\lambda_0(\theta + \frac{\pi}{4}) = \lambda_0(\frac{\pi}{4} - \theta)$ $\lambda_0(0) = 0.5$ $\lambda_0(\pi/4) = 0.25$.

We choose for λ_0 the simplest trigonometric polynomial which verifies these equalities, and property (iii) as much as possible (see appendix). So we have $\lambda_0(\theta) = 0.5 - \cos^2(\theta) + \cos^4(\theta)$ and we deduce the value of $\lambda_1(\theta)$, $\lambda_2(\theta)$, ...

We can show that it is equivalent to say that the computation of $u_{\xi\xi}$ is exact for a set of polynomials of degree as high as possible. We can include all polynomial of degree 3 plus the polynomial : $(x^2 + y^2)(x \sin\theta - y \cos\theta)^2$. (See Appendix).

Our algorithm is following:

- We estimate the gradient direction θ, and ξ the orthogonal direction using the 9 points of the stencil 3X3.
- Computation of $curv(u)$ as above.
- We proceed, in case of equation

$$\begin{array}{ll} \textbf{(4)} : & u^{n+1}(x) = u^n(x) + \Delta t u^n_{\xi\xi}(x) \\ \textbf{(1)} : & u^{n+1}(x) = u^n(x) + \Delta t (Du)^{2/3} (u^n_{\xi\xi}(x))^{1/3} \end{array}$$

In the case of (4), we must choose $\Delta t \leq 0.5(\Delta x)^2$ in order to have some stability. Whereas for (1), the maximum value of Δt cannot be imposed theoretically. This scheme verifies `[Consistency]` , because it is exact at order 2. All the geometrical properties (`[Isom. invariance]` , `[Scale invariance]` , ...) are true for all the polynomials mentionned above. For others functions, we have these properties only for the affine transforms which leave the grid invariant.

But, the scheme is not morphologic. And, it does not verify `[Order Preserving]` nor `[`L^∞ `stability]` . However, it is experimentally quite stable. In fact, we do not expect to have `[Order Preserving]` and `[Consistency]` in such a quasilinear scheme with a fixed stencil. If you impose both, as noticed in [ALM] and proved by [CrLi], you do not limit the size of the stencil.

4 Appendix.

Denoting by ξ the orthogonal direction of the gradient of u, $\nabla u curv(u)$ is equal to $u_{\xi\xi\xi}$. Suppose now that the direction ξ is known :

$$\xi = (-\sin\theta, \cos\theta)$$

has not enough geometrical invariants. For instance, a disk becomes after some time a polygon... This can be checked on the analysis of the spiral (fig. *Spiral.img*) So, *it cannot be used for sharp shape analysis because it creates corners which do not exist in the original shape.* We need more rotational invariance and this will be achieved by the next scheme.

3.2 Luis Alvarez, Frederic Guichard's scheme ([AlGu]).

In the last subsection, we tried to use an adaptative diffusion (in order to keep [L^∞ stability]), now we seek for a good estimate of the curvature and gradient direction. We want to have more isometrical invariants. Let us begin by computing a discretization of $curv(u)$ as rotational invariant as possible. We write curv(u) as a linear combination of the values of u in the 9 points.

$$(u_{\xi\xi})_{i,j} = \sum_{-1\leq l\leq 1}\sum_{-1\leq k\leq 1} \alpha_{l,k}^{i,j}(\theta_{i,j})u_{i,j} + o(\Delta x)^2$$

where $\theta_{i,j}$ is the angle of the vector ξ. At each point (i,j),we would like to define the weights $\alpha_{l,k}^{i,j}$, in order to have :

(i) the preceding equality.

(ii) the obvious geometrical invariants of the grid.

(iii) A relation which implies [L^∞ stability]

$$\sum_{-1\leq l\leq 1}\sum_{-1\leq k\leq 1} |\alpha_{l,k}^{i,j}(\theta_{i,j})| - 2 * \alpha_{0,0}^{i,j}(\theta_{i,j}) \leq 0$$

Now, if we have (i) and (ii) , we cannot have (iii) for all angles θ. Thus, we shall not have [L^∞ stability] . In order to have a maximum of stability, we change therefore (iii) into

$$\sum_{-1\leq l\leq 1}\sum_{-1\leq k\leq 1} |\alpha_{l,k}^{i,j}(\theta_{i,j})| - 2 * \alpha_{0,0}^{i,j}(\theta_{i,j}) \qquad \text{as small as possible.}$$

At last the functions $\alpha_{l,k}$ must be as smooth as possible for stability.

Properties (i), and (ii) imply that

$$\begin{cases} \alpha_{1,0} = \alpha_{-1,0} = \lambda_1(\theta) \\ \alpha_{0,1} = \alpha_{0,-1} = \lambda_2(\theta) \\ \alpha_{1,1} = \alpha_{-1,-1} = \lambda_3(\theta) \\ \alpha_{-1,1} = \alpha_{1,-1} = \lambda_4(\theta) \\ \alpha_{0,0} = -4\lambda_0(\theta) \end{cases}$$

where

$$\begin{cases} \lambda_1(\theta) = 2\lambda_0(\theta) - \sin^2\theta \\ \lambda_2(\theta) = 2\lambda_0(\theta) - \cos^2\theta \\ \lambda_3(\theta) = -\lambda_0(\theta) + 0.5(\sin\theta\cos\theta + 1) \\ \lambda_4(\theta) = -\lambda_0(\theta) - 0.5(\sin\theta\cos\theta + 1) \end{cases}$$

3 Quasilinear finite difference Schemes.

These schemes are based on a diffusive interpretation of the equation (4). Indeed, if ξ is a unit vector, such that $\nabla u.\xi = 0$, and $u_{\xi\xi}$ the second derivative of u in the direction ξ, we have :

$$(4) \Leftrightarrow \frac{\partial u}{\partial t} = u_{\xi\xi}$$

This formulation yields an anisotropic diffusion equation. Anisotropic, because we diffuse only in one direction ξ, depending on the gradient. (compare with the isotropic diffusion ($\frac{\partial u}{\partial t} = \Delta u$) i.e. the heat equation).

Equation (1) can be rewritten as $\frac{\partial u}{\partial t} = (\nabla u)^{\frac{2}{3}}(u_{\xi\xi})^{\frac{1}{3}}$, where $(\nabla u)^{\frac{2}{3}}$ is understood as the speed of the diffusion.

The schemes which we wish to discuss compute $u_{\xi\xi}$ by a quasilinear formula in the sense defined above.

3.1 Luis Alvarez's scheme for equation (4) ([ALM],[AlGu]).

The numerical scheme of Alvarez uses the main directions of the grid, given by the vectors

$$(1,0),(2,1),(1,1),(1,2),(-1,0),(-2,1),(-1,1),(-1,2)$$

At every point and every time there will be diffusion in only one of these directions. So, the algorithm is:

Step 1 : We search the nearest main direction η of ξ.

Step 2 : We assume that $u_{\xi\xi}$ is equal to $u_{\eta\eta}$. And we discretize $u_{\eta\eta}$ by

$$u_{\eta\eta}^{n+1} = \frac{u_{i-1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i+1,j}^{n+1}}{(\Delta x)^2}, \text{ if } \eta = \vec{i}$$

$$u_{\eta\eta}^{n+1} = \frac{u_{i-1,j-1}^{n+1} - 2u_{i,j}^{n+1} + u_{i+1,j+1}^{n+1}}{2(\Delta x)^2}, \text{ if } \eta = \vec{i} + \vec{j}$$

and so on... Then define an implicit scheme by

$$u^{n+1} - u^n = \Delta t u_{\eta\eta}^{n+1}$$

Step 3 : Solve the implicit scheme and return to step 1.

This scheme satisfies [Isom. invariance] for rotations of radius $\frac{\pi}{4}$, and symmetries with respect to main axis. [L^∞ stability] is also true. But it does not satisfy [Consistency] nor [Morphology] ! We remark experimentally that the scheme is robust ([L^∞ stability]). But, the code

3. The "purely morphological representation" of u by its level curves, a sequence C_k of Jordan curves discretized by affine finite elements. (Each curve is given by a sequence of vertices, with real components).

Let H_λ, R, and A the operators defined by $H_\lambda u(x) = u(\lambda x)$, $Ru(x) = u(Rx)$ where R is an isometry of $\mathbb{R}^2$, $Au(x) = u(Ax)$ where A is a linear application of $\mathbb{R}^2$. Since the schemes to be considered will be very different in structure, we cannot state all following properties in a completely formal way. What they mean in each particular case will be clear in context.

- [Consistency] : We shall say that a scheme is consistent if the discrete operator T approximating the second member of (1) or (2) tends to the differential one, when the steps Δx and Δt tend to 0 in a suitable way.
- [Convergence] : $\forall t\ T^n u \to T_t u\ a.e.$ when Δt and Δx tend to 0 in a suitable way, and $n\Delta t \to t$.
- [L^∞ stability] : Let c,d be real numbers. If c$\leq u^n(x) \leq$d then c$\leq u^{n+1}(x) \leq$d.
- [Order Preserving] : If $u^n(x) \geq v^n(x)$ for all x in $\mathbb{R}^2$, then $u^{n+1}(x) \geq v^{n+1}(x)$.
- [Morphology] : For all nondecreasing functions h : $Th(u) = h(Tu)$. This means that ***the level sets*** are handled independently.
- [Pure Morphology] : There is no interaction between the connected components of level sets. (This is a more restrictive property than [Morphology] and means that ***the level curves*** are handled independently).
- [Isom. invariance] : $TR_\theta = R_\theta T$.
- [Scale invariance] : $\forall l, n \in \mathbb{N}\ \exists n'(n,l),\ T^n H_l = H_l T^{n'}$.
- [Affine invariance] : $\forall n \in \mathbb{N}\ \exists n'(n,A),\ T^n A = A T^{n'}$ (for any matrix A with entire coefficients (only for equation 1.))
- [Quasilinear] : Let T be a discrete operator approximating a differential operator of u on a grid. (For instance, the second derivative in a direction). T is quasilinear if it can be written as a linear combination of the u_{ij}^n, where i,j belong to a fixed stencil. The coefficients of this linear combination may vary.
  We add this (not very formal) notion because quasilinear schemes are easy to implement.

In the next sections, we define schemes and discuss which of the above properties is satisfied.

"mathematical morphology" school [Ser], [Mara], associate with a picture u the set of its level sets

$$X_a u = \{x, u(x) \geq a\}$$

Then, assuming that each level set is a union of silhouettes, we can simply define $T_t(u)$ as

[morphological principle] For any a, t, h and u: $X_a T_{t+h,t}(u) = T_{t+h,t}(X_a u)$

It is easily seen [LoMo] that if we define a multiscale analysis of images from the preceding analysis of shapes by using [morphological principle], it is governed by

$$\frac{\partial u}{\partial t} = g(t, curv(u))|Du| \tag{8}$$

Conversely, this equation has a unique solution in the sense of "viscosity solutions" [AGLM]. ***So a consequence of Theorem 1 above is that there is a single image multiscale analysis, satisfying [pyramidal], [local shape inclusion], [basic principle], [affine invariance] and the [morphological principle].*** Therefore, we call the associated equation the fundamental equation of image processing

$$\frac{\partial u}{\partial t} = |Du|(curv(u))^{\frac{1}{3}}, \qquad u(0) = u_0$$

2.3 Invariance and stability requirements for numerical schemes of the fundamental equation.

In order to compare schemes, we shall fix in this section a list of formal requirements which derive from the above developed axiomatic analysis. We have first the properties which imply convergence of the schemes toward the equations, and second the invariance properties of the equations themselves. All schemes will be defined as the iteration of a nonlinear discrete filter, which we call T. We set $u^n = T^n u_0$, where u_0 is the (discretized) initial image. All schemes will depend on a scale step Δt and a space step Δx.

We shall consider three different discrete representations of u^n :

1. The classical discrete representation on a grid $u^n_{i,j} = u^n(i,j)$, with $1 \leq i \leq N$ $1 \leq j \leq N$.
2. "The morphological representation" of u by a bunch of level sets $X_a = \{u(x) \geq a\}$, where a admits, for technological reasons, 255 values.

2.1 The fundamental equation of Shape Analysis.

When a point x belongs to an evolving curve, we denote by $\dot{x}$ the time derivative of x, which is a vector of $\mathbb{R}^2$. By $curv(x)$ we denote the curvature of a curve which is C^2 at x. Recall that the curvature is defined as the inverse of the radius of the osculatory circle to the curve at x. The curvature is zero if the radius is infinite. By $\vec{n}(x)$ we denote the vector normal to the curve and pointing towards the concavity of the curve (These definitions for curves are compatible with the ones given for sets.)

Theorem 1. [LoMo]*(i) Under the three principles pyramidal, local shape inclusion and "basic", the multiscale analysis of shapes is governed by a curvature motion equation*

$$\dot{x} = g(t, curv(x))n(x) \tag{6}$$

where g is a nondecreasing function.
(ii) If we assume [Affine inv.], then the equation of the multiscale analysis is, up to rescaling:

$$\dot{x} = \gamma(t.curv(x))n(x) \tag{7}$$

where γ is defined as following: $\gamma(x) = a.x^{1/3}$ if $x \geq 0$ and $\gamma(x) = b.x^{1/3}$ if $x \leq 0$ and a, b are two nonnegative values.
(iii) If we add that $T_t(X^c) = T_t(X)^c$ [Reverse contrast invariance] then the function g in (i) is odd and $a = b$ in (ii).

By the expression "governed by", we mean that the equation must be verified at any point (t, x) of $X(t)$ where the silhouette is smooth enough to give a classical sense to both terms of the equation.

Remark: It has been proved [GaHa], [Gra], [Ang] that the equation (6) with $g(t, s) = s$ has smooth solutions. This equation has been introduced in image processing by [KTZ], [MaMo] and [AGLM1,2,3] and [Yui] in different contexts. An early version of an algorithm leading to equation (6) is in [KoDo].

The equation (7) has been first introduced and axiomatically justified in [AGLM1,2,3]. It has also been proposed in [SaTa1]. Existence and regularity of the solution are proved in [SaTa2].

2.2 The fundamental equation of Image Processing.

Since the multiscale analysis verifies the obvious inclusion principle that if $A \subset B$ then $T_{t+h,t}(A) \subset T_{t+h,t}(B)$, then we can, as well known in the

multiscale analysis makes of very simple shapes in order to specify it. So we add a basic "principle" which will state what happens to disks. As we shall see, disks are somehow the "basis" on which every shape can be decomposed.

[Basic principle] Let $D = D(x, 1/r)$ be a disk with curvature $1/r$ and center x. Then $T_{t+h,t}(D)$ is a disk with radius $\rho(t, h, 1/r)$ and center x. Moreover, the function $h \to \rho(t, h, 1/r)$ is differentiable with respect to h at $h = 0$ and the differential is continuous with respect to $1/r$.

The "basic principle" implies that the multiscale analysis behaves in a smooth and isotropic way.

In the following, we set

$$g(t, 1/r) = \frac{\partial \rho}{\partial h}(t, 0, 1/r)$$

Note that $g(t, s)$ is defined for $t \geq 0$ and $s \in I\!R$, now to define $g(t, 0)$ we must assume that $lim\ g(t, 1/r)$ exists when r tends to $+\infty$ or $-\infty$. The radius r may be positive or negative, according to the orientation of the normal $n(x)$. In the case where the curve is a Jordan curve enclosing a set X, we take as a ***convention that*** $n(x)$ is ***pointing outside*** X and ***the curvature is negative if*** X ***is convex at*** x, ***positive else.*** It may seem natural to assume therefore that $g(t, \kappa)$ is odd with respect to κ. Indeed, this corresponds to the assumption that a black disk on grey background and a white disk behave in a same way.

By the shape local inclusion principle, it is easily seen that the function $g(t, \kappa) = \frac{\partial \rho}{\partial h}(t, 0, \kappa)$ is nondecreasing with respect to κ.

The last shape preserving principle is the affine invariance of shape analysis. Set $AX(x) = X(Ax)$ for any linear map A. The affine invariance can be stated as:

[Affine inv]. There exists a C^1 function $t'(t, A) \geq 0$ defined for any **A** and $t \geq 0$, such that $AT_{t'(t,A),t'(s,A)} = T_{t,s}A$. Moreover, the function $\phi(t) = \frac{\partial t'}{\partial \lambda}(t, Id)$ is positive for $t > 0$.

This relation means that the result of the multiscale analysis T_t is independent of the size and position in space of the analyzed features. This is very important in the world where we live, since the same plane object X can be seen at very different distances and orientations and therefore with any affine change $X(x) \to X(Ax)$. **A** linear map corresponds to the anamorphosis of a plane image when it is presented to the eye at an arbitrary distance and with an arbitrary orientation in space.

essential for establishing requirements for numerical schemes and in order to justify the conjectures.

Define a *shape or silhouette* as a closed set X whose boundary is a Jordan curve of $\mathbb{R}^2$. We denote by $T_t(X)$ the *shape analysed at scale* t. X is identified with its characteristic function $X(x) = 1$ if $x \in X$ and 0 else. We call multiscale analysis any family operators $(T_t)_{t\geq 0}$ acting on shapes and we set $X(t) = T_t(X)$.

We first consider a principle which is called in vision theory the *pyramidal architecture* property. According to this assumption, T_t can be computed from T_s for any $s \leq t$ and T_0 is of course the identity. This is natural, since a coarser analysis of the original picture is likely to be deduced from a finer one without any dependence upon the original picture. Of course the finest picture analysis is the identity. In order to formalize the pyramidal architecture we include $T_t = T_{t,0}$ into a family of *transition operators* $T_{t,s}$ indexed by $0 \leq s \leq t$ and satisfying

[Pyramidal architecture] $T_{t+s} = T_{t+s,s}T_s$ **for all** $0 \leq s$, $0 \leq t$

The shape analysis process is thought of as a flow of the shape through different filters, each associated with a scale t. If we discretize such a process, it yields a serie of operators $S_n = T_{(n+1)h,nh}$ applied successively.

Assume that X and Y are two silhouettes and that for some $x \in \partial Y$ and some $r > 0$, one has $X \cap B(x, 2r) \subset Y \cap B(x, 2r)$. Assume further that the inclusion is strict in the sense that ∂X and ∂Y only meet eventually at x. Then we shall say that *the shape X is included in shape Y around x*.

[Shape local inclusion] If X is included in Y around x, then for h small enough, $T_{t+h,t}(X) \cap B(x, r) \subset T_{t+h,t}(Y) \cap B(x, r)$.

This last axiom implies that the value of $T_{t+h,t}(X)$ for h small, at any point x, is determined by the behaviour of X near x. We are allowed to take r infinite. Therefore the shape local inclusion principle also implies that if a shape is globally contained in another, this order is preserved for every scale [MaMo]. Notice that the combination of both pyramidal and shape local inclusion principles is generally called in Image Processing *causality* [Koe], [Wit], [Kim], [SaTa], [KTZ]. Indeed, the causality assumption in Image Processing states that when the scale increases, no new structure appears in the image. Hummel [Hum] noticed that the causality assumption can be reduced to maximum principle.

Both preceding principles allow, as we shall see, to localize the shape analysis in space and time and we therefore only need to state what the

uniqueness results and implies the equivalence of both approaches ([EvSp], [CGG] ...). Therefore, (at least) two kinds of schemes are available for the applications in image processing :*we can either identify a picture with the set of its level curves and then evolve them jointly by using the curve evolution framework. We also can evolve the image itself by using the PDE formulation.* We shall introduce a third numerical approach, inspired from the *Mathematical Morphology* [Serra]. This approach consists in considering the level sets of the image $X_a = \{u(x) \geq a\}$ and to handle them by operations which we call "affine openings", "affine closings", "affine median filters", etc.... We shall see that the iteration of these filters should tend theoretically towards equation (1). Before starting with the discussion, let us mention that many ideas for the schemes which we discuss here come from the image processing litterature, but they were in many cases not correctly implemented or interpreted *because the subjacent PDE models where not identified.* We shall make this assertion more precise in the next sections.

In section 2, we shall start with an explanation of the use of the fundamental equation in image processing. This explanation is necessary, because it leads to ask for new kinds of (long time) invariance properties when defining numerical schemes. These invariance properties are a key to applications, e.g. in shape recognition. Therefore, we shall recall an axiomatic characterization of equation (2) which was developed in [LoMo], simplifying the founding approach of [AGLM1]. This axiomatization leads us to a list of invariance requirements for numerical schemes which will be the basis of our discussion through the paper.

In section 3, we present and discuss classical finite difference schemes.

In section 4, we relate the fundamental equation to a new "affine morphological" set theory and derive new schemes for these equations.

In section 5, we discuss curve evolution schemes and show why they can be understood as special implementations of morphological schemes.

Section 6 is devoted to the experimental comparison of the numerical schemes and to the precise statement and justification of several conjectures about their convergence.

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2 The four principles of Multiscale Shape Analysis and the fundamental equation.

Let us start with a list of four principles of shape analysis which lead to the fundamental equation (For more details, see [AGLM], [LoMo]). These principles will be

1 Introduction.

In the following, we consider an image $u_0(x)$ defined on the plane and with real values representing the brightness. We also consider smooth Jordan curves $C(s)$ where s denotes either the euclidean parameter, defined up to a shift by the relation $|C_s| = 1$, or the affine parameter, defined by $|det(C_s, C_{ss})| = 1$ ([SaTa]). We shall discuss some numerical schemes for five equations in $\mathbb{R}^2$ related with image processing.

The most important equation to be considered is the fundamental equation of image processing

$$\frac{\partial u}{\partial t} = |Du|(curv(u))^{\frac{1}{3}}, \qquad u(0) = u_0 \tag{1}$$

It can be interpreted as the motion of the level curves of u by the intrinsic equation

$$\frac{\partial C}{\partial t} = C_{ss} \tag{2}$$

where s is the affine parameter. Indeed, this last equation can be rewritten [SaTa2] as

$$\frac{\partial C}{\partial t} = curv(C(s))^{1/3}\vec{n}(s) \tag{3}$$

where $\vec{n}(s)$ is the normal at s to the curve $C(s)$ and $curv(C(s))$ is the curvature of C at s.

In order to make the discussion more fruitful, we shall also include in the discussion the euclidian analogs of these equations, that is

$$\frac{\partial u}{\partial t} = |Du|curv(u), \qquad u(0) = u_0 \tag{4}$$

$$\frac{\partial C}{\partial t} = C_{ss} = curv(C(s))\vec{n}(s) \tag{5}$$

where s is the euclidian parameter. Indeed, at a point $x = C(s)$ where u is smooth and $Du \neq 0$, one has

$$curv(u).\frac{Du}{|Du|} = curv(C(s))\vec{n}(s) = C_{ss}$$

where C denotes the level curve of u passing by x. (What s is, length or affine parameter, will be clear in context).

Equations (2) and (5) have been solved, with existence, regularity and uniqueness results when the initial Jordan curve is smooth in [Grayson, Gage-Hamilton, Sapiro-Tannenbaum]. A general theory has been made for equations (1) to (5) by using the viscosity solution framework [AGLM-1-2-3]. It also yields existence and

Numerical Analysis of the fundamental equation of image processing.

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Abstract

The fundamental equation of image processing

$$\frac{\partial u}{\partial t} = |Du|(curv(u))^{\frac{1}{3}}$$

has been discovered recently with existence proofs in the sense of viscosity solutions [AGLM-1-2-3]. Another approach, by modelling curve evolution has been introduced in [SaTa] and yields the intrinsic equation $\frac{\partial C}{\partial t} = C_{ss}$ where s is the "affine length" along the curve. The second equation describes the motion of the level curves of a solution u of the first equation; therefore both models are equivalent. We introduce a third reformulation for this equation, coming from the "mathematical morphology" set analysis. This diversity of formulations leads to a rich discussion about numerical schemes: classical finite differences, curve evolution schemes, and set evolution schemes. We compare the numerical results and discuss the new kind of problems arising in their numerical analysis: Indeed, the fundamental equation has been created for achieving multiscale shape analysis. Therefore, geometric, affine and scale invariance properties should be very accurately satisfied. We discuss a nonlocal non monotonic scheme for the equation which achieves all invariance properties. We also state (and justify) several convergence conjectures for the new introduced schemes and discuss the experimental results obtained on several shapes. They led to a crossvalidation of two very different schemes.