Chapter 11

Contrast-Invariant Classes of Functions and Their Level Sets

This chapter is about one of the major technological contributions of mathematical morphology, namely the representation of images by their upper level sets. As we shall see in this chapter, this leads to a handy contrast invariant representation of images.

Definition 11.1. *Let $u \in \mathcal{F}$. The level set of u at level $0 \leq \lambda \leq 1$ is denoted by $\mathcal{X}_\lambda u$ and defined by*

$$\mathcal{X}_\lambda u = \{\mathbf{x} \mid u(\mathbf{x}) \geq \lambda\}.$$

Strictly speaking, we have called level sets what should more properly be called upper level sets. Several level sets of a digital image are shown in Figure 11.1 and all of the level sets of a synthetic image are illustrated in Figure 11.2. The reconstruction of an image from its level sets is illustrated in Figure 11.3. Two important properties of the level sets of a function follow directly from the definition. The first is that the level sets provide a complete description of the function. Indeed, we can reconstruct u from its level sets $\mathcal{X}_\lambda u$ by the formula

$$u(\mathbf{x}) = \sup\{\lambda \mid \mathbf{x} \in \mathcal{X}_\lambda u\}.$$

This formula is called *superposition principle* as u is being reconstructed by "superposing" its level sets.

Exercise 11.1. Prove the superposition principle. ■

The second important property is that level sets of a function are globally invariant under contrast changes. We say that two functions u and v have the same level sets globally if for every λ there is μ such that $\mathcal{X}_\mu v = \mathcal{X}_\lambda u$, and conversely. Now suppose that a contrast change $g : \mathbb{R} \to \mathbb{R}$ is continuous and increasing. Then it is not difficult to show that $v = g(u)$ and u have the same level sets globally.

Exercise 11.2. Check this last statement for any function u and any continuous increasing contrast change g. ■

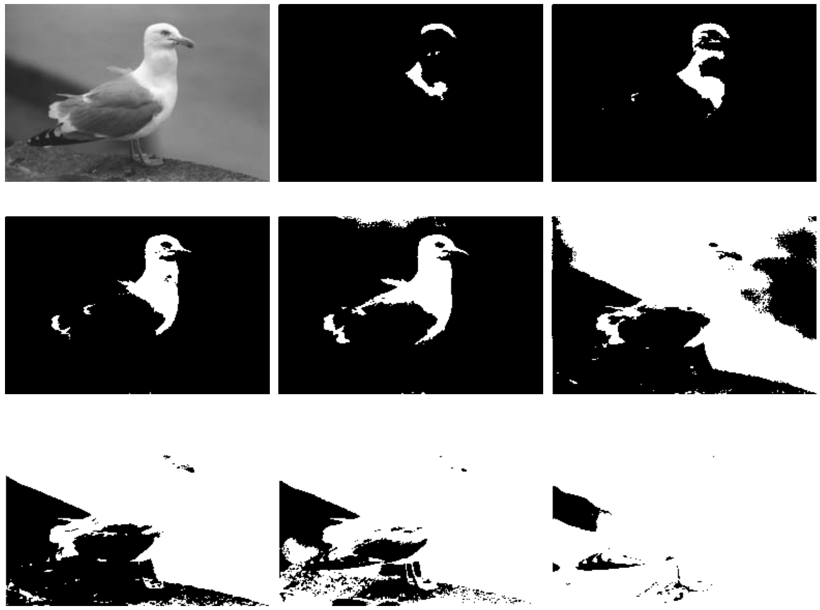

Figure 11.1: Level sets of a digital image. Left to right, top to bottom: We first show an image with range of gray levels from 0 to 255. Then we show eight level sets in decreasing order from $\lambda = 225$ to $\lambda = 50$, where the grayscale step is 25. Notice how essential features of the shapes are contained in the boundaries of level sets, the level lines. Each level set (which appears as white) is contained in the next one, as guaranteed by Proposition 11.2.

Conversely, we shall prove that if the level sets of a function $v \in \mathcal{F}$ are level sets of u, then there is a continuous contrast change g such that $v = g(u)$. This justifies the attention we will dedicate to level sets, as they turn out to contain all of the contrast invariant information about u.

11.1 From an image to its level sets and back

In the next proposition, for a sake of generality, we consider bounded measurable functions on S_N, not just functions in $\mathcal{F}$.

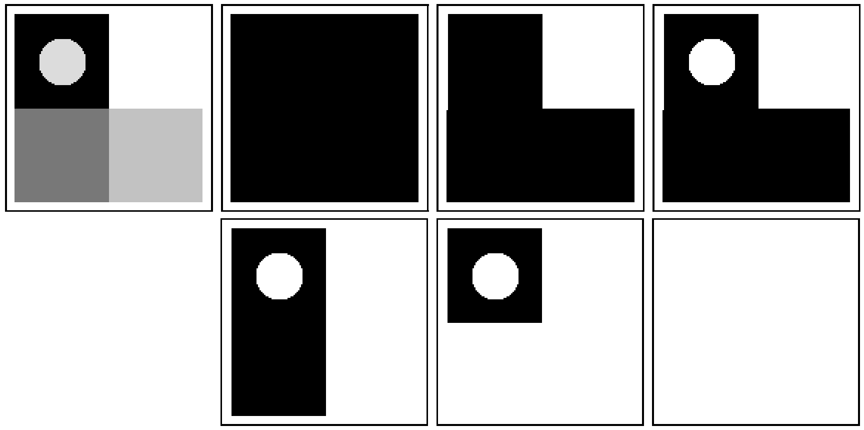

Figure 11.2: A simple synthetic image and all of its level sets (in white) with decreasing levels, from left to right and from top to bottom.

Proposition 11.2. *Let X_λ denote the level sets $\mathcal{X}_\lambda u$ of a bounded measurable function $u : S_N \to \mathbb{R}$. Then the sets X_λ satisfy the following two structural properties:*

(*i*) *If $\lambda > \mu$, then $X_\lambda \subset X_\mu$. In addition, there are two real numbers $\lambda_{max} \geq \lambda_{min}$ so that $X_\lambda = S_N$ for $\lambda < \lambda_{min}$, $X_\lambda = \emptyset$ for $\lambda > \lambda_{max}$.*

(*ii*) *$X_\lambda = \bigcap_{\mu<\lambda} X_\mu$ for every $\lambda \in \mathbb{R}$.*

Conversely, if $(X_\lambda)_{\lambda\in\mathbb{R}}$ is a family of sets of $\mathcal{M}$ that satisfies (i) and (ii), then the level sets of the function u defined by superposition principle,

$$u(\mathbf{x}) = \sup\{\lambda \mid \mathbf{x} \in X_\lambda\} \tag{11.1}$$

satisfy $\mathcal{X}_\lambda u = X_\lambda$ for all $\lambda \in \mathbb{R}$ and $\lambda_{min} \leq u \leq \lambda_{max}$.

Proof. The first part of Relation (*i*) follows directly from the definition of upper level sets. The second part of (*i*) works with $\lambda_{min} = \inf u$ and $\lambda_{max} = \sup u$. The relation (*ii*) follows from the equivalence $u(\mathbf{x}) \geq \lambda \Leftrightarrow u(\mathbf{x}) \geq \mu$ for every $\mu < \lambda$.

Conversely, take a family of subsets $(X_\lambda)_{\lambda\in\mathbb{R}}$ satisfying (*i*) and (*ii*) and define u by the superposition principle. Let us show that $X_\lambda = \mathcal{X}_\lambda u$. Take first $\mathbf{x} \in X_\lambda$. Then it follows from the definition of u that $u(\mathbf{x}) \geq \lambda$, and hence $\mathbf{x} \in \mathcal{X}_\lambda u$. Thus, $X_\lambda \subset \mathcal{X}_\lambda u$. Conversely, let $\mathbf{x} \in \mathcal{X}_\lambda u$. Then $u(\mathbf{x}) = \sup\{\nu \mid \mathbf{x} \in X_\nu\} \geq \lambda$. Consider any $\mu < \lambda$. Then there exists a μ' such that $\mu < \mu' \leq \sup\{\nu \mid \mathbf{x} \in X_\nu\}$ and $\mathbf{x} \in X_{\mu'}$. It follows from (*i*) that $\mathbf{x} \in X_\mu$. Since μ was any number less that λ, we conclude by using (*ii*) that $\mathbf{x} \in \bigcap_{\mu<\lambda} X_\mu = X_\lambda$. It is easily checked that $\lambda_{min} \leq u \leq \lambda_{max}$. □

Exercise 11.3. Check the last statement of the preceding proof, that $\lambda_{min} \leq u \leq \lambda_{max}$. ■

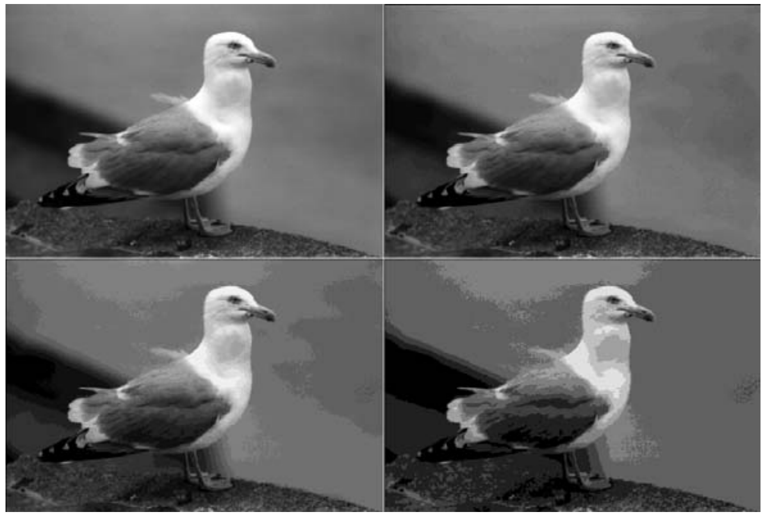

Figure 11.3: Reconstruction of an image from its level sets: an illustration of Proposition 3.2. We use four different subsets of the image's level sets to give four reconstructions. Top, left: all level sets; top, right: all level sets whose gray level is a multiple of 8; bottom, left: multiples of 16; bottom, right: multiples of 32. Notice the relative stability of the image shape content under these drastic quantizations of the gray levels.

11.2 Contrast changes and level sets

Practical aspects of contrast changes are illustrated in Figures 11.4, 11.5, 11.6, and 11.7, which illustrate how insensitive our perception of images is to contrast changes, even when they are flat on some interval. When this happens, some information on the image is even lost, as several grey levels melt together.

Definition 11.3. *Any nondecreasing continuous surjection $g: \mathbb{R} \to \mathbb{R}$ will be called a contrast change.*

Exercise 11.4. Remark that $g(s) \to \pm\infty$ as $s \to \pm\infty$. Check that if $u \in \mathcal{F}$ and g is a contrast change, then $g(u) \in \mathcal{F}$. ■

In case g is increasing, g has an inverse contrast change g^{-1}. In case g is flat on some interval, we shall be happy with a pseudo-inverse for g.

Definition 11.4. *The pseudo-inverse of any contrast change $g : \mathbb{R} \to \mathbb{R}$ is defined by*

$$g^{(-1)}(\lambda) = \inf\{r \in \mathbb{R} \mid g(r) \geq \lambda\}.$$

Exercise 11.5. Check that g^{-1} is finite on $\mathbb{R}$ and tends to $\pm\infty$ as $s \to \pm\infty$. Give an example of g such that g^{-1} is not continuous. ■

Exercise 11.6. Compute and draw $g^{(-1)}$ for the function $g(s) = \max(0, s)$. Notice that such a function is ruled out by our conditions at infinity for contrast changes. ■

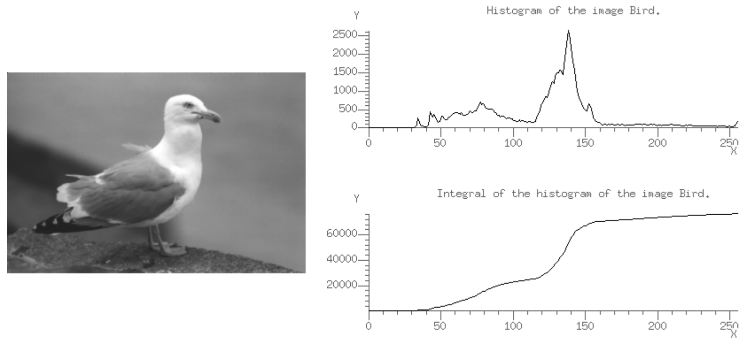

Figure 11.4: The histogram of the image Bird. For each $i \in \{0, 1, \ldots, 255\}$, we display (above, right) the function $h(i) = \text{Card}\,\{\mathbf{x} \mid u(\mathbf{x}) = i\}$. The function below is the cumulative histogram, namely the primitive of h defined by $H_u(i) = \text{Card}\,\{\mathbf{x} \mid u(\mathbf{x}) \le i\}$. The shape of h provides an indication about the overall contrast of the image and about the contrast change imposed by the sensors. See Chap. 12 for manipulations of the cumulative histogram.

Lemma 11.5. *Let $g : \mathbb{R} \to \mathbb{R}$ be a contrast change. Then for every $\lambda \in \mathbb{R}$, $g(g^{(-1)})(\lambda) = \lambda$ and*

$$g(s) \ge \lambda \text{ if and only if } s \ge g^{(-1)}(\lambda). \tag{11.2}$$

Proof. The first relation follows immediately from the continuity of g. If $g(s) \ge \lambda$, then $s \ge g^{(-1)}(\lambda)$ by the definition of $g^{(-1)}(\lambda)$. Conversely, if $s \ge g^{(-1)}(\lambda)$, then $g(s) \ge g(g^{(-1)}(\lambda)) = \lambda$ and thus $g(s) \ge \lambda$. □

Theorem 11.6. *Let $u \in \mathcal{F}$ and g be a contrast change. Then any level set of $g(u)$ is a level set of u. More precisely, for $\lambda \in \mathbb{R}$,*

$$\mathcal{X}_\lambda g(u) = \mathcal{X}_{g^{(-1)}(\lambda)} u. \tag{11.3}$$

Proof. The proof is read directly from Lemma 11.5 by taking $s = u$. □

The next result is a converse statement to Theorem 11.6.

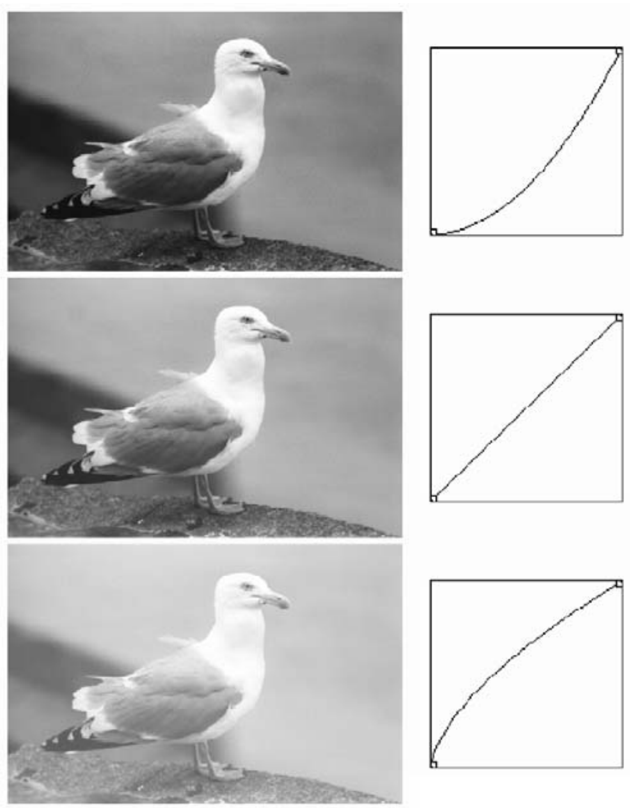

Figure 11.5: Contrast changes and an equivalence class of images. The three images have exactly the same level sets and level lines, but their level sets are mapped onto three different gray-level scales. The graphs on the right are the graphs of the contrast changes $u \mapsto g(u)$ that have been applied to the initial gray levels. The first one is concave; it enhances the darker parts of the image. The second one is the identity; it leaves the image unaltered. The third one is convex; it enhances the brighter parts of the image. Software allows one to manipulate the contrast of an image to obtain the best visualization. From the image analysis viewpoint, image data should be considered as an equivalence class under all possible contrast changes.

Theorem 11.7. *Let u and $v \in \mathcal{F}$ such that every level set of v is a level set of u. Then $v = g(u)$ for some contrast change g.*

Proof. One can actually give an explicit formula for g, namely, for every $\mu \in u(S_N)$,

$$g(\mu) = \sup\{\lambda \in v(S_N) \mid \mathcal{X}_\mu u \subset \mathcal{X}_\lambda v\}. \tag{11.4}$$

For $\mu \notin u(S_N)$, we can easily extend g into an nondecreasing function such that $g(\pm\infty) = \pm\infty$). (Take (e.g.) g piecewise affine). Note that $\nu > \mu$ implies that $g(\nu) \geq g(\mu)$. Let us first show that $\inf v \leq g(\mu) \leq \sup v$. Set

$$\Lambda := \{\lambda \mid \mathcal{X}_\mu u \subset \mathcal{X}_\lambda v\}.$$

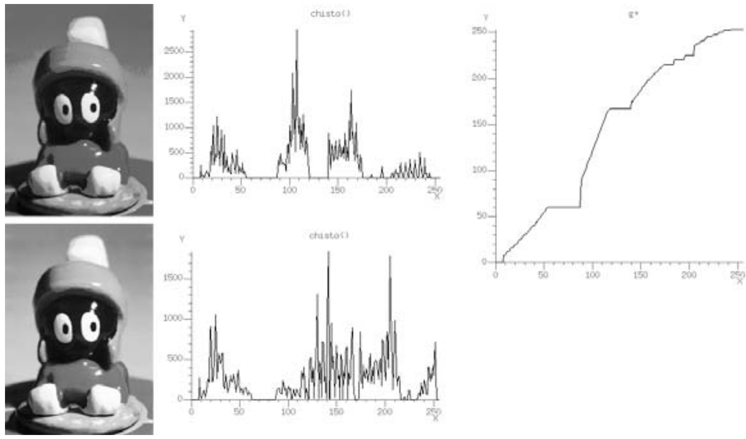

Figure 11.6: The two images (left) have the same set of level sets. The contrast change that maps the upper image onto the lower image is displayed on the right. It corresponds to one of the possible g functions whose existence is stated in Corollary 3.14. The function g may be locally constant on intervals where the histogram of the upper image is zero (see top, middle graph). Indeed, on such intervals, the level sets are invariant.

Λ is not empty because $\mathcal{X}_{\inf v} = S_N$ and therefore $\inf v \in \Lambda$. Thus $g(\mu) = \sup \Lambda \geq \inf v$. On the other hand $\mathcal{X}_{\sup v+\varepsilon} v = \emptyset$ for every $\varepsilon > 0$. Since $\mu \in u(S_N)$, $\mathcal{X}_\mu u \neq \emptyset$ and therefore $g(\mu) = \sup \Lambda \leq \sup v$.

Step 1: Proof that $v(\mathbf{x}) \geq g(u(\mathbf{x}))$. By Proposition 11.2($i$) Λ has the form $(-\infty, \sup \Lambda)$ or $(-\infty, \sup \Lambda]$. But by Proposition 11.2(ii), $\mathcal{X}_{\sup \Lambda} v = \bigcap_{\lambda < \sup \Lambda} \mathcal{X}_\lambda v$, and this implies by the definition of Λ that $g(\mu) = \sup \Lambda \in \Lambda$. Thus,

$$\mathcal{X}_\mu u \subset \mathcal{X}_{g(\mu)} v. \tag{11.5}$$

Given $\mathbf{x} \in S_N$, let $\mu = u(\mathbf{x})$ in (11.5). Then,

$$\mathcal{X}_{u(\mathbf{x})} u \subset \mathcal{X}_{g(u(\mathbf{x}))} v.$$

Since $\mathbf{x} \in \mathcal{X}_{u(\mathbf{x})} u$, we conclude that $\mathbf{x} \in \mathcal{X}_{g(u(\mathbf{x}))} v = \{\mathbf{y} \mid v(\mathbf{y}) \geq g(u(\mathbf{x}))\}$.

Step 2: Proof that $v(\mathbf{x}) \leq g(u(\mathbf{x}))$. Given $\mathbf{x} \in S_N$, we translate the assumption with $\lambda = v(\mathbf{x})$ as follows: There exists a $\mu(\mathbf{x}) \in \mathbb{R}$ such that

$$\mathcal{X}_{v(\mathbf{x})} v = \{\mathbf{y} \mid u(\mathbf{y}) \geq \mu(\mathbf{x})\} = \mathcal{X}_{\mu(\mathbf{x})} u. \tag{11.6}$$

Since $\mathbf{x} \in \mathcal{X}_{v(\mathbf{x})} v$, we know that $\mathbf{x} \in \mathcal{X}_{\mu(\mathbf{x})} u$. Thus, $u(\mathbf{x}) \geq \mu(\mathbf{x})$, and $\mathcal{X}_{u(\mathbf{x})} u \subset \mathcal{X}_{\mu(\mathbf{x})} u = \mathcal{X}_{v(\mathbf{x})} v$. This last relation implies by the definition of g that $v(\mathbf{x}) \leq g(u(\mathbf{x}))$.

Step 3: Proof that g is continuous. Recall that the image of a connected set by a continuous function is connected. Thus $u(S_N)$ is an interval of $\mathbb{R}$ and so is $v(S_N)$. Since $g(u) = v$, $g(u(S_N)) = v(S_N)$ is an interval. Now, a nondecreasing function is continuous on an interval if and only if its range is connected. Thus g is continuous on $u(S_N)$ and so is its extension to $\mathbb{R}$. □

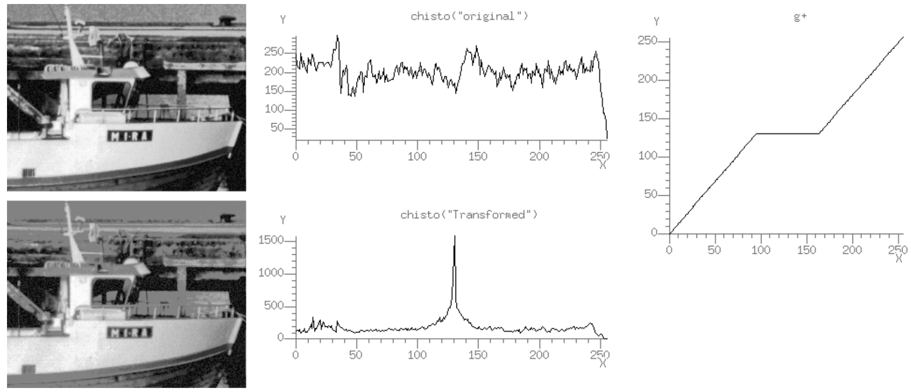

Figure 11.7: The original image (top, left) has a strictly positive histogram (all gray levels between 0 and 255 are represented). Therefore, if any contrast change g that is not strictly increasing is applied, then some data will be lost. Every level set of the transformed image $g(u)$ is a level set of the original image; however, the original image has more level sets than the transformed image.

Exercise 11.7. Prove the last statement in the theorem, namely that "a nondecreasing function is continuous on an interval if and only if its range is connected". ■

Exercise 11.8. By reading carefully the steps 1 and 2 of the proof of Theorem 11.7, check that this theorem applies with u and v just bounded and measurable on S_N. Then one has still has $v = g(u)$ with g defined in the same way. Of course g is still nondecreasing but not necessarily continuous. Find a simple example of functions u and v such that g is not continuous. ■

11.3 Exercises

Exercise 11.9. This exercise gives a way to compute the function g such that $v = g(u)$ defined in the proof of Theorem 11.7 in terms of the repartition functions of u and v. Let G be a Gauss function defined on $\mathbb{R}^N$ such that $\int_{\mathbb{R}^N} G(\mathbf{x})d\mathbf{x} = 1$. For every measurable subset of $\mathbb{R}^N$, set $|A|_G := \int_A G(\mathbf{x})d\mathbf{x}$. Let u be a bounded continuous function on $\mathbb{R}^N$. We can associate with u its *repartition function* $h_u(\lambda) := |\mathcal{X}_\lambda u|_G$. Show that $h_u : \lambda \in [\inf u, \sup u] \to h_u(\lambda)$ is strictly decreasing. Show that it can have jumps but is left-continuous, that is $h_u(\lambda) = \lim_{\mu\uparrow\lambda} h_u(\mu)$. Define for every non increasing function h a pseudo inverse by $h^{((-1))}(\mu) := \sup\{\lambda \mid h(\lambda) \geq \mu\}$. Show that $h^{((-1))}$ is non increasing and that $h^{((-1))} \circ h(\mu) \geq \mu$, and that if h is left-continuous, $h \circ h^{((-1))}(\mu) \geq \mu$. Using (11.4) prove that $g = h_v^{((-1))} \circ h_u$.
Hint: prove that $g(\mu) = \sup\{\lambda \mid |\mathcal{X}_\mu u|_G \leq |\mathcal{X}_\lambda v|_G\}$. ■

Exercise 11.10. Let u be a real-valued function. If $(\mu_n)_{n\in\mathbb{N}}$ is an increasing sequence that tends to λ, prove that

$$\mathcal{X}_\lambda u = \bigcap_{n\in\mathbb{N}} \mathcal{X}_{\mu_n} u \tag{11.7}$$

$$\{\mathbf{x} \mid u(\mathbf{x}) > \lambda\} = \bigcup_{\mu>\lambda} \mathcal{X}_\mu u. \tag{11.8}$$

■

11.4 Comments and references

Contrast invariance and level sets. It was Wertheimer who noticed that the actual local values of the gray levels in an image could not be relevant information for the human visual system [164]. Contrast invariance is one of the fundamental model assumptions in mathematical morphology. The two basic books on this subject are Matheron [107] and Serra [144, 146]. See also the fundamental paper by Serra [145]. Ballester et al. defined an "image intersection" whose principle is to keep all pieces of bilevel sets common to two images [12]. (A bilevel set is of the form $\{\mathbf{x} \mid \lambda \leq u(\mathbf{x}) \leq \mu\}$.) Monasse and Guichard developed a *fast level set transform* (FLST) to associate with every image the inclusion tree of connected components of level sets [119]. They show that the inclusion trees of connected upper and lower level sets can be fused into a single inclusion tree; among other applications, this tree can be used for image registration. See Monasse [118].

Contrast changes. The ability to vary the contrast (to apply a contrast change) of a digital image is a very useful tool for improving image visualization. Professional image processing software has this capability, and it is also found in popular software for manipulating digital images. For more about contrast changes that preserve level sets, see [30]. Many reference on contrast-invariant operators are given at the end of Chapter **??**.

Chapter 12

Specifying the contrast of images

Midway image equalization means any method giving to a pair of images a similar histogram, while maintaining as much as possible their previous grey level dynamics. The comparison of two images is one of the main goals of computer vision. The pair can be a stereo pair, two images of the same object (a painting for example), multi-channel images of the same region, images of a movie, etc. Image comparison is perceptually greatly improved if both images have the same grey level dynamics (which means, the same grey level histogram). Many image comparison algorithms are based on grey level and take as basic assumption that intensities of corresponding points in both images are equal. However, this assumption is generally false for stereo pairs, and deviations from this assumption cannot even be modeled by affine transforms [36]. Consequently, if we want to compare visually and numerically two images, it is useful to give them first the same dynamic range and luminance.

In all of this applicative chapter the images $u(\mathbf{x})$ and $v(\mathbf{x})$ are defined on a domain which is the union of M pixels. The area of each pixel is equal to 1. The images are discrete in space and values: they attain values in a finite set $\mathbb{L}$ and they are constant on each pixel of the domain. We shall call such images *discrete images*. The piecewise constant interpolation is a very bad image interpolation. It is only used here for a fast handling of image histograms. For other scopes, better interpolation methods are of course necessary.

Definition 12.1. *Let u be a discrete image. We call cumulative histogram of u the function $H_u : \mathbb{L} \to \mathbb{M} := [0, M] \cap \mathbb{N}$ defined by*

$$H_u(l) =: \text{meas}(\{\mathbf{x} \mid u(\mathbf{x}) \leq l\}).$$

This cumulative histogram is a primitive of the *histogram* of the image $h(l) = \text{meas}(\{\mathbf{x} \mid u(\mathbf{x}) = l\})$. Figures 11.4, 11.6 and the first line of Figure 12.1. show the histograms of some images and their cumulative histograms. In fact Figure 11.7 shows first the histogram and then the modified histogram after a contrast change has been applied. These experiments illustrate the robustness of image relevant information to contrast changes and even to the removal of some level sets, when the contrast change is flat on an interval. Such experiments suggest that one can *specify* the histogram of a given image by applying the adequate contrast change. Before proceeding, we have to define the pseudo-inverses of a discrete function.

Proposition 12.2. *Let $\varphi: \mathbb{L} \to \mathbb{M}$ be a nondecreasing function from a finite set of values into another. Define two pseudo-inverse functions for φ :*

$$\varphi^{(-1)}(l) := \inf\{s \mid \varphi(s) \geq l\} \text{ and } \varphi^{((-1))}(l) := \sup\{s \mid \varphi(s) \leq l.\}$$

Then one has the following equivalences:

$$\varphi(s) \geq l \Leftrightarrow s \geq \varphi^{(-1)}(l), \quad \varphi(s) \leq l \Leftrightarrow s \leq \varphi^{((-1))}(l) \tag{12.1}$$

and the identity

$$(\varphi^{(-1)})^{((-1))} = \varphi. \tag{12.2}$$

Proof. The implication $\varphi(s) \geq l \Rightarrow s \geq \varphi^{(-1)}(l)$ is just the definition of $\varphi^{(-1)}$. The converse implication is due to the fact that the infimum on a a finite set is attained. Thus $\varphi(\varphi^{(-1)}(l)) \geq l$ and therefore $s \geq \varphi^{(-1)}(l) \Rightarrow \varphi(s) \geq l$. The identity (12.2) is a direct consequence of the equivalences (12.1). Indeed,

$$s \leq (\varphi^{(-1)})^{((-1))}(l) \Leftrightarrow \varphi^{(-1)}(s) \leq l \Leftrightarrow s \leq \varphi(l).$$

□

Exercise 12.1. Prove that if φ is increasing, $\varphi^{(-1)} \circ \varphi(l) = l$ and $\varphi^{((-1))} \circ \varphi(l) = l$. If φ is surjective, $\varphi \circ \varphi^{(-1)} = l$ and $\varphi \circ \varphi^{((-1))}(l) = l$. ■

Proposition 12.3. *Let φ be a discrete contrast change and u a digital image. Then*

$$H_{\varphi(u)} = H_u \circ \varphi^{((-1))}.$$

Proof. By (12.1), $\varphi(u) \leq l \Leftrightarrow u \leq \varphi^{((-1))}(l)$. Thus by the definitions of H_u and $H_{\varphi(u)}$,

$$H_{\varphi(u)}(l) = \text{meas}(\{\mathbf{x} \mid \varphi(u) \leq l\}) = \text{meas}(\{\mathbf{x} \mid u(\mathbf{x}) \leq \varphi^{((-1))}(l)\}) = H_u \circ \varphi^{((-1))}(l).$$

□

Let $G: \mathbb{L} \to \mathbb{M} := [0, 1, \dots, M]$ be any discrete nondecreasing function. Can we find a contrast change $\varphi: \mathbb{L} \to \mathbb{L}$ such that the cumulative histogram of $\varphi(u)$, $H_{\varphi(u)}$ becomes equal to G? Not quite: if for instance u is constant its cumulative histogram is a one step function and Proposition 12.3 implies that $H_{\varphi(u)}$ will also be a one step function. More generally if u attains k values, then $\varphi(u)$ attains k values or less. Hence its cumulative histogram is a step function with $k+1$ steps. Yet, at least formally, the functional equation given by Proposition 12.3, $H_u \circ \varphi^{-1} = G$, leads to $\varphi = G^{-1} \circ H_u$. We know that we cannot get true inverses but we can involve pseudo-inverses. Thus, we are led to the following definition:

Definition 12.4. *Let $G : \mathbb{L} \to \mathbb{M}$ be a nondecreasing function. We call specification of u on the cumulative histogram G the image*

$$\tilde{u} := G^{((-1))} \circ H_u(u).$$

Exercise 12.2. Prove that if G and H_u are one to one, then the cumulative histogram of $\tilde{u}$ is G. Is it enough to assume that H_u is one to one? ∎

Definition 12.5. *Let, for $l \in [0, L] \cap I\!N$, $G(l) = \lfloor \frac{M}{L} l \rfloor$, where $\lfloor r \rfloor$ denotes the largest integer smaller than r. Then $\tilde{u} := G^{((-1))} \circ H_u(u)$ is called the* uniform equalization of u. *If v is another discrete image and one takes $G = H_v$, $\tilde{u} := H_v^{((-1))} \circ H_u(u)$ is called the* specification of u on v.

When H_u is one to one, one can reach by applying a contrast change to u any specified cumulative histogram G. Otherwise, the above definitions do the best that can be expected and are actually quite efficient. For instance in the "marshland experiment" (Figure 12.1) the equalized histogram and its cumulative histogram are displayed on the second row. The cumulative histogram is very close to its goal, the linear function. The equalized histogram does not look flat but a sliding average of it would look almost flat.

Yet it is quite dangerous to specify the histogram of an image with an arbitrary histogram specification. This fact is illustrated in Figures 12.1 and 12.2 where a uniform equalization erases existing textures by making them too flat (Figure 12.1) but also enhances the quantization noise in low contrasted regions and produces artificial edges or textures (see Figure 12.2).

12.1 Midway equalization

We have seen that if one specifies u on v, then u inherits roughly the histogram of v. It is sometimes more adequate to bring the cumulative histograms of u and v towards a cumulative histogram which would be "midway" between both. Indeed, if we want to compare visually and numerically two images, it is useful to give them first the same dynamic range and luminance. Thus we wish:

- From two images u and v, construct by contrast changes two images $\tilde{u}$ and $\tilde{v}$, which have a similar cumulative histogram.
- This common cumulative histogram h should stand "midway" between the previous cumulative histograms of u and v, and be as close as possible to each of them. This treatment must avoid to favor one cumulative histogram rather than the other.

Definition 12.6. *Let u and v be two discrete images. Set*

$$\Phi := \frac{1}{2}\left(H_u^{(-1)} + H_v^{(-1)}\right).$$

We call midway cumulative histogram *of u and v the function*

$$G := \Phi^{((-1))} = \left(\frac{1}{2}(H_u^{(-1)} + H_v^{(-1)})\right)^{((-1))} \tag{12.3}$$

and "midway specifications" of u and v the functions $\tilde{u} := \Phi \circ H_u(u)$ and $\tilde{v} := \Phi \circ H_v(v)$.

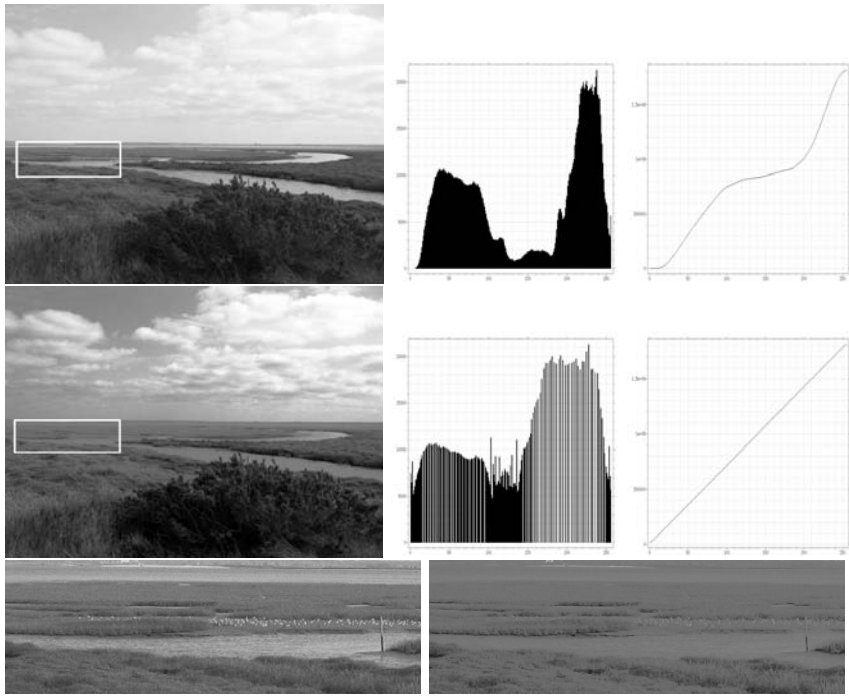

Figure 12.1: *First row: Image u, the corresponding grey level histogram h_u, and the cumulative histogram H_u. Second row: Equalized image $H_u(u)$, its histogram and its cumulative histogram. In the discrete case, histogram equalization flattens the histogram as much as possible. We see on this example that image equalization can be visually harmful. In this marshland image, after equalization, the water is no more distinguishable from the vegetation. The third row shows a zoom on the rectangular zone, before and after equalization.*

Exercise 12.3. Let u and v be two constant images, whose values are a and b. Prove that their "midway" function is the right one, namely a function w which is constant and equal to $\frac{a+b}{2}$. ■

Exercise 12.4. Prove that if we take as a definition of the midway histogram

$$G := \left(\frac{1}{2}(H_u^{((-1))} + H_v^{((-1))})\right)^{(-1)},$$

then for two constant images $u = a$ and $v = b$ the midway image is constant and equal to $[1/2(a+b) - 1]$. This proves that Definition 12.6 is better. ■

Exercise 12.5. Prove that if u is a discrete image and f and g two nondecreasing functions, then the midway image of $f(u)$ and $g(u)$ is $\frac{f(u)+g(u)}{2}$. ■

Exercise 12.6. If we want the "midway" cumulative histogram H to be a compromise between H_u and H_v, the most elementary function that we could imagine is their average, which amounts to average their histograms as well. However, the following example proves that this idea is not judicious at all.

Consider two images whose histograms are "crenel" functions on two disjoint intervals, for instance $u(\mathbf{x}) := ax$, $v(\mathbf{x}) = bx + c$. Compute a, b, c in such a way that h_u and h_v have disjoint supports. Then compute the specifications of u and v on the

Figure 12.2: *Effect of histogram equalization on the quantization noise. On the left, the original image. On the right, the same image after histogram equalization. The effect of this equalization on the dark areas (the piano, the left part of the wall), which are low contrasted, is perceptually dramatic. We see many more details but the quantization noise has been exceedingly amplified.*

mean cumulative histogram $G := \frac{H_u+H_v}{2}$. Compare with their specifications on the midway cumulative histogram. ■

12.2 Midway equalization on image pairs

Results on a stereo pair

The top of Figure 12.3 shows a pair of aerial images in the region of Toulouse. Although the angle variation between both views is small, and the photographs are taken at nearly the same time, we see that the lightning conditions vary significantly (the radiometric differences can also come from a change in camera settings). The second line shows the result of the specification of the histogram of each image on the other one. The third line shows both images after equalization.

If we scan some image details, as illustrated on Figure 12.4, the damages caused by a direct specification become obvious. Let us specify the darker image on the brightest one. Then the information loss, due to the reduction of dynamic range, can be detected in the brightest areas. Look at the roof of the bright building in the top left corner of the image (first line of Figure 12.4): the chimneys project horizontal shadows on the roof. In the specified image, these shadows have almost completely vanished, and we cannot even discern the presence of a chimney anymore. In the same image after equalization, the shadows are still entirely recognizable, and their size reduction remains minimal. The second line of Figure 12.4 illustrates the same phenomenon, observed in the bottom center of the image. The structure present at the bottom of the image has completely disappeared after specification and remains visible after midway equalization. These examples show how visual information can be lost by specification and how midway algorithms reduce significantly this loss.

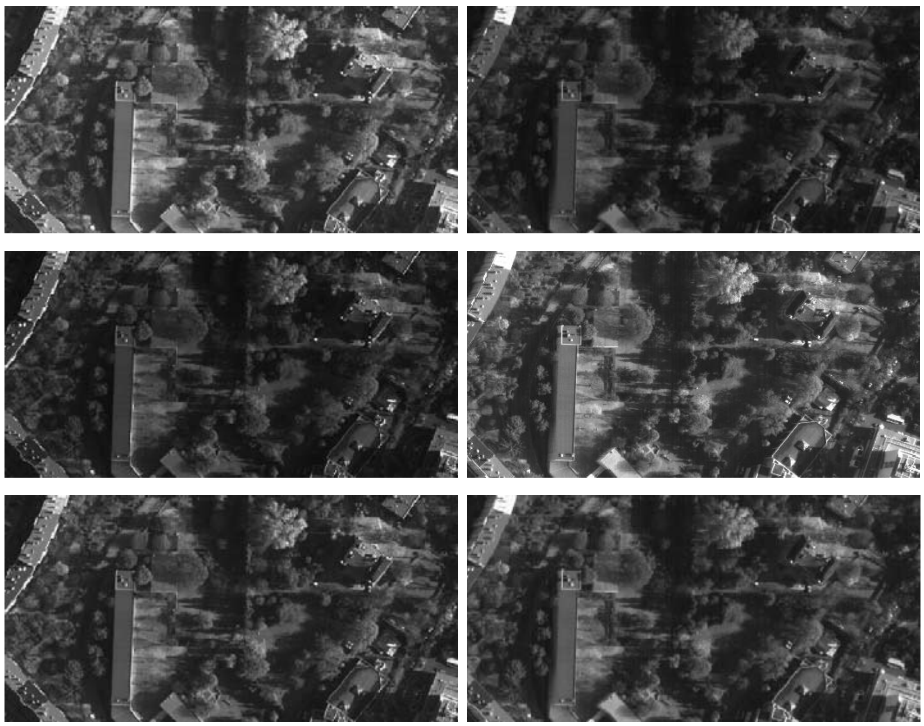

Figure 12.3: Stereo pair: two pieces of aerial images of a region of Toulouse. Same images after specification of their histograms on each other (left: the histogram of the first image has been specified on the second, and right: the histogram of the second image has been specified on the first). Stereo pair after midway equalization.

Figure 12.4: *Extracts from the stereo pair shown on Figure 12.3. From left to right: in the original image, in the specified one, in the original image after midway equalization. Notice that no detail is lost in the midway image, in contrast with the middle image.*

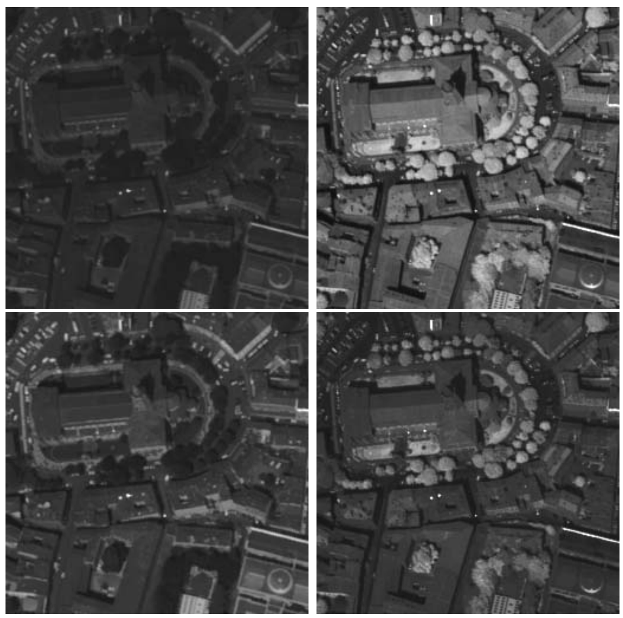

Figure 12.5: *First line: two images of Toulouse (blue and infrared channel). Second line: same images after midway equalization.*

Multi-Channel images

The top of Figure 12.5 shows two pieces of multi-channel images of Toulouse. The first one is extracted from the blue channel, and the other one from the infrared channel. The second and third line of the same figure show the same images after midway equalization. The multichannel images have the peculiarity to present contrast inversions : for instance, the trees appear to be darker than the church in the blue channel, and are naturally brighter than the church in the infrared channel. The midway equalization being limited to increasing contrast changes, it obviously cannot handle these contrast inversions. In spite of these contrast inversions, the results remain visually good, which underlines the robustness of the method gives globally a good equalization.

Photographs of the same painting

The top of Figure 12.6 shows two different snapshots of the same painting, *Le Radeau de la Méduse*[1], by Théodore Géricault (small web public versions). The second one is brighter and seems to be damaged at the bottom left. The second line shows the same couple after midway equalization. Finally, the last line of Figure 12.6 shows the difference between both images after equalization. We see clear differences around the edges, due to the fact that the original images are not completely similar from the geometric point of view.

12.2.1 Movie equalization

One can define a midway cumulative histogram to an arbitrary number of images. This is extremely useful for the removal of flicker in old movies. Flicker has multiple causes, physical, chemical or numerical. The overall contrast of successive images of the same scene in a movie oscillates, some images being dark and others bright. Our main assumption is that image level sets are globally preserved from one image to the next, even if their level evolves. This leads to the adoption of a movie equalization method preserving globally all level sets of each image. We deduce from Theorem 11.7 in the previous chapter that the correction must be a global contrast change on each image. Thus the only left problem is to *specify* a common cumulative histogram (and therefore a common histogram) to all images of a given movie scene. Noticing that the definition of G in (12.3) for two images simply derives from a mean, its generalization is easy. Let us denote $u(t,\mathbf{x})$ the movie (now a discrete time variable has been added) and by H^t the cumulative histogram function of $\mathbf{x} \to u(t,\mathbf{x})$ at time t. Since flicker is localized in time, the idea is to define a time dependent cumulative histogram function K_t^h which will be the "midway" cumulative histogram of the cumulative histograms in an interval $[t-h, t+h]$. Of course the linear scale space theory of Chapter 3 applies here. The ideal average is gaussian. Hence the following definition.

Definition 12.7. *Let $u(t,\mathbf{x})$ be a movie and denote by H_t the cumulative histogram of $u(t) : \mathbf{x} \to u(t,\mathbf{x})$. Consider a discrete version of the 1-D gaussian $G_h(t) = \frac{1}{(4\pi h)^{\frac{1}{2}}} e^{-\frac{t^2}{4h}}$. Set*

$$\Phi_{(t,l)} := \int G_h(t-s)(H_s^{(-1)})(l)ds.$$

We call "midway gaussian cumulative histogram at scale h" of the movie $u(t,\mathbf{x})$ the time dependent cumulative histogram

$$\mathbb{G}_{(t,l)} := \Phi_{(t,l)}^{((-1))} = \left(\int G_h(t-s)(H_s^{(-1)})(l)ds \right)^{((-1))} \tag{12.4}$$

and "midway specification" of the movie $u(t)$ the function $\tilde{u}(t) := \Phi \circ H_{u(t)}(u(t))$. If $H_{u(t)}$ is surjective, then $\tilde{u}(t)$ has $G_{(t,l)}$ as common cumulative histogram.

Notice that this is a straightforward extension of Definition 12.6.

[1] Muse du Louvre, Paris.

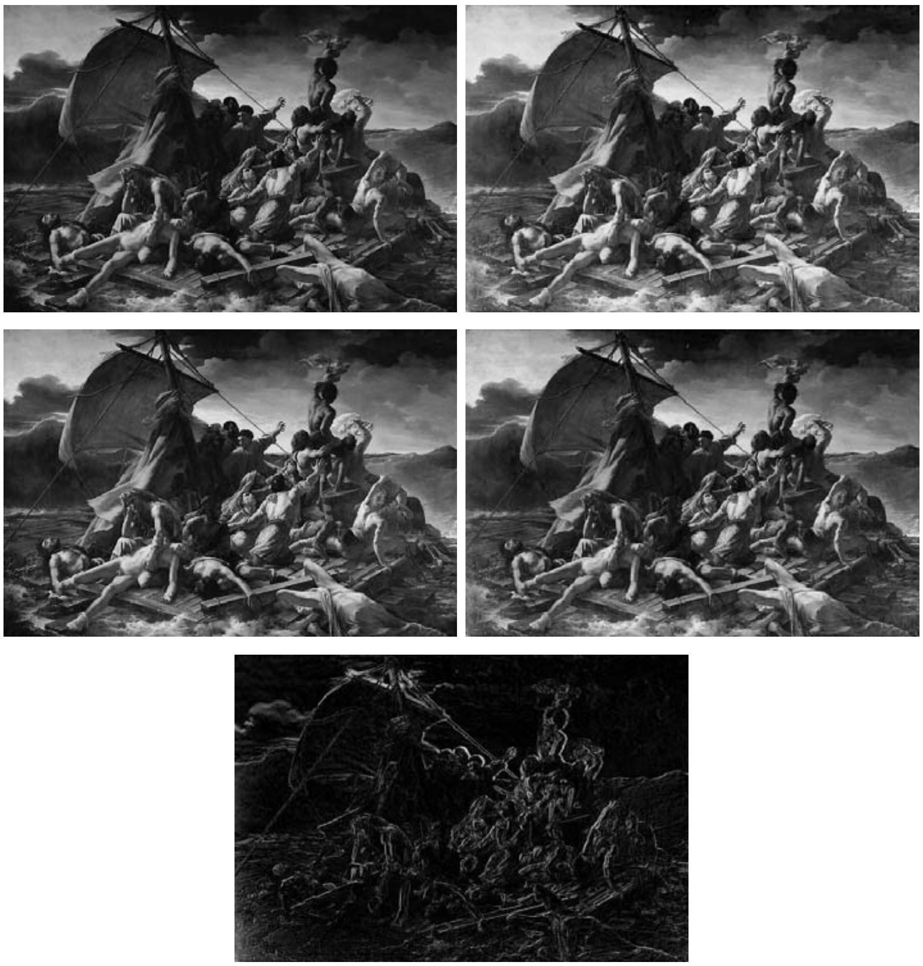

Figure 12.6: Two shots of the *Radeau de la Méduse, by Géricault*. The same images after midway equalization. Image of the difference between both images after equalization. The boundaries appearing in the difference are mainly due to the small geometric distortions between the initial images.

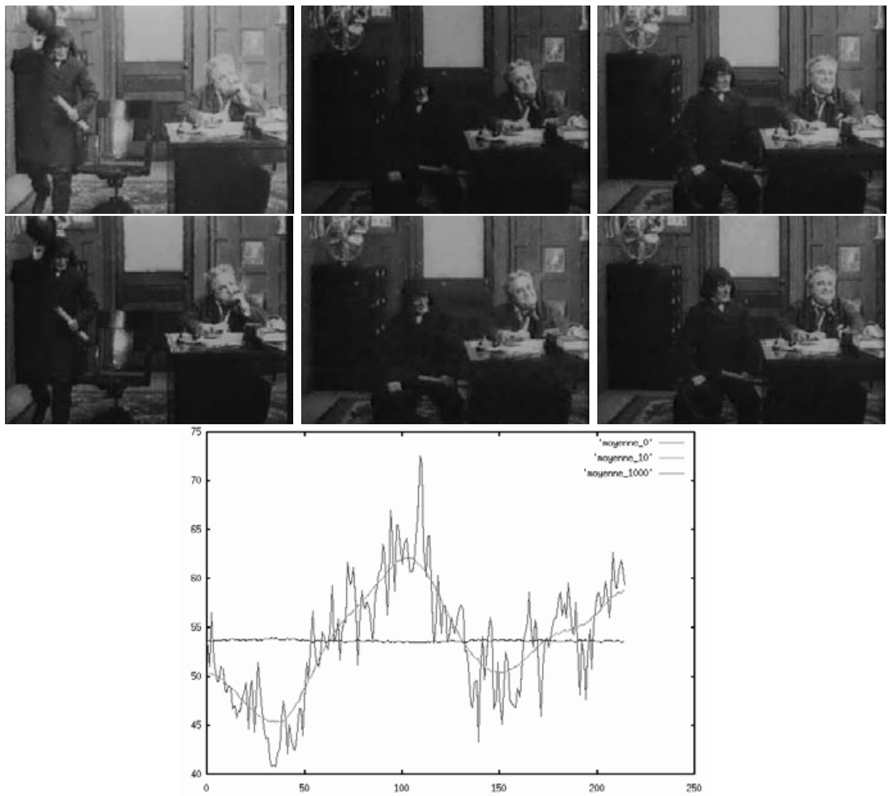

Figure 12.7: *(a) Three images of Chaplin' s film His New Job, taken at equal intervals of time. This extract of the film suffers from a severe real flicker. (b) Same images after the Scale-Time Equalization at scale $s = 100$. The flicker observed before has globally decreased. (c) Evolution of the mean of the current frame in time and at three different scales. The most oscillating line is the mean of the original sequence. The second one is the mean at scale $s = 10$. The last one, almost constant, corresponds to the large scale $s = 1000$. As expected the mean function is smoothed by the heat equation.*

The implementation and experimentation is easy. We simply show in Figure 12.7 three images of Chaplin' s film *His New Job*, taken at equal intervals of time. This extract of the film suffers from a severe real flicker. This flicker is corrected at the scale where, after gaussian midway equalization, the image mean becomes nearly constant through the sequence. The effects of this equalization are usually excellent. They are easily extended to color movies by processing each channel independently.

12.3 Comments and references

Histogram specification As we have seen *histogram specification* [60] can be judicious if both images have the same kind of dynamic range. For the same reason as in equalization, this method can also product contouring artifacts. The midway theory is essentially based on Julie Delons' PhD and papers [40], [41] where she defines two midway histogram interpolation methods. One of them, the *square root* method involves the definition of a square root of any nondecreasing function g, namely a function g such that $f \circ f = g$. Assume that u and v come from the same image (this intermediate image is unknown), up to two contrast changes f and f^{-1}. The function f is unknown, but satisfies formally the equality $H_u \circ f = H_v \circ f^{-1}$. Thus

$$H_u{}^{-1} \circ H_v = f \circ f.$$

It follows that the general method consists in building an increasing function f such that $f \circ f = H_u{}^{-1} \circ H_v$ and replacing v by $f(v)$ and u by $f^{-1}(u)$. This led Delon [?] to call this new histogram midway method, the "square root" equalization. The midway interpolation developed in this chapter uses mainly J. Delon's second definition of the midway cumulative histogram as the harmonic mean of the cumulative histograms of both images. This definition is preferable to the square root. Indeed, both definitions yield very similar results but the harmonic mean extends easily to an arbitrary number of images and in particular to movies [41]. The Cox, Roy and Hingorani algorithm defined in [36] performs a midway equalization. They called their algorithm "Dynamic histogram warping" and its aim is to give a common cumulative histogram (and therefore a common histogram) to a pair of images. Although their method is presented as a dynamic algorithm, there is a very simple underlying formula, which is the harmonic mean of cumulative histograms discovered by Delon [40].