Proof of Comparison Principle: Theorem 3.1

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1 The main result

Let T be a positive number, $T \in I\!R$, and let us consider a degenerate parabolic equation of the form

$$u_t = F(t, x, D^2 u) \qquad \text{in} \qquad Q = (0, T] \times I\!R^n. \tag{1}$$

We list the assumptions on $F = F(t, x, X)$ which are necessary for the result given in **Theorem 3.1**.

(F1) $F : Q \times S_n \to I\!R$ is continuous.

(F2) F is degenerate elliptic, i.e.,

$$F(t, x, X) \leqslant F(t, x, X + Y) \qquad \forall Y \geqslant 0.$$

(F3) For every $R > 0$

$$c_R = \sup\{|F(t, x, X)| : \quad ||X|| \leqslant R, (t, x, X) \in Q \times S_n\} < \infty.$$

(F4) Suppose that

$$-\mu \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leqslant \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leqslant \nu \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \omega \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{2}$$

with $\mu, \nu, \omega \geqslant 0$. Then it holds:

$$F(t, y, -Y) - F(t, x, X) \geqslant -m(\nu ||x - y||^2),$$

for some modulus m independent of $t, x, y, X, Y, \mu, \nu, \omega$.

Then, the main result we will prove is the following

Theorem 1.1 *Suppose that F satisfies (F1)-(F4). Let u and v be respectively, sub and supersolutions of (1). Assume that*

(i) *$u(t,x) \leqslant K(||x||+1)$, $v(t,x) \geqslant -K(||x||+1)$ for some $K>0$ independent of $(t,x) \in Q$.*

(ii) *$u(0,x) - v(0,y) \leqslant K||x-y||$ for all $(x,y) \in I\!R^n \times I\!R^n$, for some $K > 0$ independent of (x,y).*

Then there is a modulus m such that

$$u(t,x) - v(t,y) \leqslant m(||x-y||) \quad \text{on } U = (0,T] \times I\!R^n \times I\!R^n. \tag{3}$$

In particular $u \leqslant v$ on Q.

2 Preliminary results

In this section we give a list of results necessary to prove Theorem 1.1. We shall give the proof of some of them. In the other cases, we will refer to [3].

Proposition 1 *Suppose F satisfies (F1)-(F3). Let u and v be, respectively, viscosity sub and supersolutions of (1) in Q. Assume that u and v satisfy (i) and (ii). Then for $K' > K$, there is a constant $M = M(K', F) > 0$ such that*

$$u(x,t) - v(y,t) \leqslant K'||x-y|| + M(1+t) \quad \text{on} \quad U = I\!R^n \times I\!R^n \times (0,T]. \tag{4}$$

Proof. We set

$$w(x,y,t,s) = u(x,t) - v(y,s),$$
$$\phi(x,y,t) = K'(||x-y||^2+1)^{1/2} + M(1+t).$$

We will prove that

$$w(x,y,t,t) \leqslant \phi(x,y,t) \tag{5}$$

for $(x,y,t) \in U$ and M sufficiently large.

Let $\{g_R\}_{R>0}$ be a family of non-negative $\mathcal{C}^2$ functions, satisfying

(g1) $g_R(x) = 0$ for $||x|| < R$,

(g2) $\frac{g_R(x)}{||x||} \to 1$ as $||x|| \to \infty$,

(g3) $G = \sup\{||\nabla g_R(x)|| + ||D^2 g_R(x)|| : \; x \in I\!R^n, \; R > 0\}$ is finite.

We set $\varphi = \phi + 2K'g_R$. By (i) and (g2), we have that for R_1 sufficiently large,

$$w(x,y,t,s) - \varphi(x,y,t) < 0 \text{ if } ||x||^2 + ||y||^2 \geqslant R_1^2, \quad 0 \leqslant t,s \leqslant T. \tag{6}$$

By (ii), we see that

$$w(x,y,0,0) - \varphi(x,y,0) < 0 \quad (x,y) \in I\!R^n \times I\!R^n. \tag{7}$$

Now, we consider for $\delta > 0$, sufficiently small,

$$\Psi(x, y, t, s) = \varphi(x, y, t) + \frac{(t-s)^2}{\delta}$$

and suppose that (4) were false, i.e., there exists $(\bar{x}, \bar{y}, \bar{t}) \in U$ such that $w(\bar{x}, \bar{y}, \bar{t}, \bar{t}) > \phi(\bar{x}, \bar{y}, \bar{t})$. For R sufficiently large,

$$w(\bar{x}, \bar{y}, \bar{t}, \bar{t}) - \Psi(\bar{x}, \bar{y}, \bar{t}, \bar{t}) > 0.$$

It follows that

$$\sup_{\bar{V}}(w - \Psi) > 0 \tag{8}$$

with $V = U \times (0, T]$. By $(6) - (8)$ and since w is upper semicontinuous, we observe that $w - \Psi$ attains a maximum over $\bar{V}$ at a point $(\hat{x}, \hat{y}, \hat{t}, \hat{s}) \in V$. This implies that

$$(\partial_t \Psi, \nabla_x \Psi, D_x^2 \Psi)(\hat{x}, \hat{y}, \hat{t}, \hat{s}) \in \mathcal{P}_Q^{2,+} u(\hat{x}, \hat{t}),$$

$$(-\partial_s \Psi, -\nabla_x y \Psi, -D_y^2 \Psi)(\hat{x}, \hat{y}, \hat{t}, \hat{s}) \in \mathcal{P}_Q^{2,+} v(\hat{y}, \hat{s}),$$

Since u and v are, respectively, viscosity sub and supersolutions of (1), we see that

$$\partial_t \Psi + F(\hat{x}, \hat{t}, D_x^2 \varphi) \leqslant 0 \tag{9}$$

$$-\partial_s \Psi + F(\hat{y}, \hat{s}, -D_y^2 \varphi) \geqslant 0. \tag{10}$$

By (g3) and the definition of ϕ, we have that $||D^2\varphi|| \leqslant N$, with $N = N(K', G)$. Subtracting (9) from (10) and by using (F3), we obtain

$$\partial_t \Psi + \partial_s \Psi \leqslant 2c_N.$$

By the other hand, $\partial_t \Psi = \partial_t \varphi + \frac{2}{\delta}(t-s)$, $\partial_s \Psi = -\frac{2}{\delta}(t-s)$ and $\partial_t \varphi = M$, which implies $M \leqslant 2c_N$. If M is taken larger than $2c_N$, we have a contradiction and (5) is proved. Finally, to estimate (4), we replace M by $M + K'$ and it follows from (5). $\square$

For $\epsilon, \delta, \gamma > 0$ we set

$$\Phi(x, y, t) = w(x, y, t) - \Psi(x, y, t),$$

$$w(x, y, t) = u(x, t) - v(y, t),$$

$$\Psi(x, y, t) = \frac{||x-y||^4}{4\epsilon} + B(x, y, t),$$

$$B(x, y, t) = \delta(||x||^2 + ||y||^2) + \frac{\gamma}{T-t}.$$

The next proposition is the same as Proposition 2.4 in [3] and we refer to it for the proof.

Proposition 2 *Suppose that* u *and* v *satisfy (4) and that*

$$\alpha = \lim_{\theta \downarrow 0} \sup\{w(x,y,t) : \ ||x-y|| < \theta, \ (x,y,t) \in \bar{U}\} > 0. \tag{11}$$

Then there are positive constants δ_0 *and* γ_0 *such that*

$$\sup_{\bar{U}} \Phi(x,y,t) > \frac{\alpha}{2} \tag{12}$$

holds for all $0 < \delta < \delta_0$, $0 < \gamma < \gamma_0$, $\epsilon > 0$.

The next proposition follows in the same manner as Proposition 2.5 of [3].

Proposition 3 *Let* u, v, δ_0, γ_0 *be as in Proposition 2. Suppose that* w *is upper semi-continuous in* $\bar{U}$.

(i) Φ *attains a maximum over* $\bar{U}$ *at* $(\hat{x}, \hat{y}, \hat{t}) \in \bar{U}$ *with* $\hat{t} < T$.

(ii) $||\hat{x} - \hat{y}||$ *is bounded as a function of* $0 < \epsilon < 1$, $0 < \delta < \delta_0$, $0 < \gamma < \gamma_0$.

(iii) $\delta\hat{x}$ *and* $\delta\hat{y}$ *tend to zero as* $\delta \to 0$*; the convergence is uniform in* $0 < \epsilon < 1$ *and* $0 < \gamma < \gamma_0$. *In particular, for fixed* $\delta > 0$, $\hat{x}$ *and* $\hat{y}$ *are bounded on* $0 < \epsilon < 1$, $0 < \gamma < \gamma_0$.

(iv) $||\hat{x} - \hat{y}||$ *tends to zero as* $\epsilon \to 0$*; the convergence is uniform in* $0 < \delta < \delta_0$ *and* $0 < \gamma < \gamma_0$.

Proposition 4 *Assume the hypotheses of Proposition 3 hold. Suppose that hypothesis (ii) of Theorem 1.1 holds for* u *and* v. *Then there is* $\epsilon_0 > 0$ *such that* Φ *attains a maximum over* $\bar{U}$ *at an interior point* $(\hat{x}, \hat{y}, \hat{t}) \in I\!R^n \times I\!R^n \times (0,T)$ *for all* $0 < \epsilon < \epsilon_0$, $0 < \delta < \delta_0$ *and* $0 < \gamma < \gamma_0$.

Proof. Suppose that the conclusion is false. Since $\hat{t} < T$, by Proposition 3, there exits sequences $\{\epsilon_i\}$ with $\epsilon_i \to 0$, $\{\delta_i\} \subset (0, \delta_0)$ and $\{\gamma_i\} \subset (0, \gamma_0)$ such that $(\hat{x}_i, \hat{y}_i, 0)$ is a maximum of Φ for $\epsilon = \epsilon_i$, $\delta = \delta_i$ and $\gamma = \gamma_i$. By (4) and (ii) of Theorem 1.1 we see

$$\frac{\alpha}{2} \leqslant \Phi(\hat{x}_i, \hat{y}_i, 0) \leqslant u(\hat{x}_i, 0) - v(\hat{y}_i, 0) \leqslant K||\hat{x}_i - \hat{y}_i||$$

Since $\epsilon_i \to 0$, applying Proposition 3 (iv) yields $||\hat{x}_i - \hat{y}_i|| \to 0$ which leads to a contradiction since $\alpha > 0$. □

The next proposition follows in the same manner as Proposition 4.4 of [3].

Proposition 5 *Suppose that* u *and* v *satisfy (4) and that expression (11) holds. Let* $(\hat{x}, \hat{y}, \hat{t})$ *be as in Proposition 3. Then*

$$\lim_{\epsilon \downarrow 0} \overline{\lim_{\delta, \gamma \downarrow 0}} \frac{||\hat{x} - \hat{y}||^4}{\epsilon} = 0 \tag{13}$$

holds.

3 Proof of Theorem 1.1

The basic idea of the proof of Theorem 1.1 is similar to that of Proposition 1. Here Ishii's idea plays an important role. We use the following Lemma proved in [2].

Lemma 1 *([2]) Let u_i be an upper semicontinuous function with $u_i < \infty$ in $I\!R_i^N \times (0,T)$ for $i = 1, 2, \ldots, k$. Let w be a function in $I\!R^N \times (0,T)$ given by*

$$w(x,t) = u_1(x,t) + \cdots + u_k(x,t)$$

for $x = (x_1, \ldots, x_k) \in I\!R^N$, where $N = N_1 + \cdots + N_k$. For $s \in (0,T)$, $z \in I\!R^N$ suppose that

$$(\tau, p, A) \in \mathcal{P}^{2,+} w(z,s)$$

Assume that there exists $\omega > 0$ such that for every $M > 0$

$$\sigma_i \leqslant C \quad \textit{whenever } (\sigma_i, q_i, Y_i) \in \mathcal{P}^{2,+} u_i(x_i, t),$$

$$||x_i - z_i|| + |s - t| < \omega \textit{ and } |u_i(x_i,t)| + ||q_i|| + ||Y_i|| \leqslant M \quad (i = 1, \ldots, k),$$

with some $C_C(M)$. Then for each $\lambda > 0$ there exists $(\tau_i, X_i) \in I\!R \times S^{N_i}$ such that

$$(\tau_i, p_i, X_i) \in \bar{\mathcal{P}}^{2,+} u_i(z_i, s) \quad i = 1, \ldots, k,$$

$$-\left(\frac{1}{\lambda} + ||A||\right) I \leqslant \begin{pmatrix} X_1 & \ldots & O \\ \vdots & & \vdots \\ O & \ldots & X_k \end{pmatrix} \leqslant A + \lambda A^2$$

and

$$\tau_1 + \cdots + \tau_k = \tau,$$

where I denotes the identity matrix and $p = (p_1, \ldots, p_k)$.

Proof of Theorem 1.1

We may assume that equation (1) has a form

$$u_t + u = F(x, t, u, D^2 u) \tag{14}$$

with the property

(F5') $\quad r \to F(x,t,r,X)$ is nonincreasing for all $(x,t,x,X) \in I\!R^n \times (0,T) \times I\!R \times S^n$,

(stronger than (F5)) if we replace u (resp. v) by $e^{\lambda t} u$ (resp. $e^{\lambda t} v$) with sufficiently large λ. The other assumptions on F are unaltered by this transformation and also hold for (14).

We argue by a contradiction. Suppose that (3) were false, then α in (11) were positive and by Proposition 1, we can apply all the conclusions of Propositions 2-5 to Φ defined above. Proposition 4 says that Φ attains a maximum over $\bar{U}$ at $(\hat{x},\hat{y},\hat{t}) \in I\!R^n \times I\!R^n \times (0,T)$ for small ϵ,δ,γ. In particular

$$w(x,y,t) \leqslant w(\hat{x},\hat{y},\hat{t}) + \Psi(x,y,t) - \Psi(\hat{x},\hat{y},\hat{t}) \quad \text{in } U.$$

Expanding Ψ at $(\hat{x},\hat{y},\hat{t})$ yields

$$(\Psi_t, \nabla\Psi, A)(\hat{x},\hat{y},\hat{t}) \in \mathcal{P}^{2,+} w(\hat{x},\hat{y},\hat{t}) \quad \text{with } D^2\Psi(\hat{x},\hat{y},\hat{t}) \leqslant A. \tag{15}$$

Applying Lemma 1 with $u_1 = u$, $u_2 = -v$, $s = \hat{t}$ and $z = (\hat{x},\hat{y})$, we conclude that for each $\lambda > 0$ there are (τ_1, X) and $(\tau_2, Y) \in I\!R \times S^n$ such that

$$(\tau_1, \nabla_x \hat{\Psi}, X) \in \bar{\mathcal{P}}^{2,+} u(\hat{x},\hat{t}),$$

$$(-\tau_2, -\nabla_y \hat{\Psi}, -Y) \in \bar{\mathcal{P}}^{2,-} v(\hat{y},\hat{t}), \tag{16}$$

$$\hat{\Psi}_t = \tau_1 + \tau_2,$$

$$-\left(\frac{1}{\lambda} + ||A||\right) I \leqslant \begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leqslant A + \lambda A^2, \tag{17}$$

where $\hat{\Psi}_t = \Psi_t(\hat{x},\hat{y},\hat{t})$, $\nabla_x \hat{\Psi} = \nabla_x \Psi(\hat{x},\hat{y},\hat{t})$, etc. Since u and v are, respectively, sub- and supersolutions of (14), it follows from (16) that

$$\tau_1 + \hat{u} - F(\hat{x},\hat{t},\hat{u},X) \leqslant 0, \qquad -\tau_2 + \hat{v} - F(\hat{y},\hat{t},\hat{v},-Y) \geqslant 0$$

subtracting yields

$$\hat{\Psi}_t + \hat{u} - \hat{v} + F(\hat{y},\hat{t},\hat{v},-Y) - F(\hat{x},\hat{t},\hat{u},X) \leqslant 0.$$

By the monotonicity property (F5'), using $\hat{\Psi}_t \geqslant \gamma T^{-2}$ and by the fact that our assumption implies $\hat{u} - \hat{v} > \alpha/2$, the last inequality becomes

$$\frac{\alpha}{2} + F(\hat{y},\hat{t},\hat{u},-Y) - F(\hat{x},\hat{t},\hat{u},X) < 0. \tag{18}$$

Differentiating Ψ and denoting by $\eta = \hat{x} - \hat{y}$, we have

$$\hat{\Psi}_x = \frac{||\eta||^2\eta}{\epsilon} + 2\delta\hat{x}, \quad \hat{\Psi}_y = -\frac{||\eta||^2\eta}{\epsilon} + 2\delta\hat{y}, \tag{19}$$

and

$$\begin{aligned} \begin{pmatrix} \hat{\Psi}_{xx} & \hat{\Psi}_{xy} \\ \hat{\Psi}_{yx} & \hat{\Psi}_{yy} \end{pmatrix} &= \tfrac{1}{\epsilon}(||\eta||^2 + 2\eta \otimes \eta) \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\delta \begin{pmatrix} I & O \\ O & I \end{pmatrix} \\ &\leqslant \tfrac{3}{\epsilon}||\eta||^2 \begin{pmatrix} I & -I \\ I & -I \end{pmatrix} + 2\delta \begin{pmatrix} I & O \\ O & I \end{pmatrix} = A \end{aligned} \tag{20}$$

With the choice of this matrice A, (17) becomes

$$-\mu\begin{pmatrix} I & O \\ O & I \end{pmatrix} \leqslant \begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leqslant \nu\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \omega\begin{pmatrix} I & O \\ O & I \end{pmatrix} \tag{21}$$

with

$$\mu = \lambda^{-1} + \frac{6||\eta||^2}{\epsilon} + 2\delta,$$

$$\nu = (\frac{3}{\epsilon} + \frac{18\lambda}{\epsilon^2}||\eta||^2 + \frac{12\delta\lambda}{\epsilon})||\eta||^2$$

and

$$\omega = 4\delta^2\lambda + 2\delta.$$

We will study the inequality (18). Fix ϵ, γ such that $0 < \epsilon < \epsilon_0$, $0 < \gamma < \gamma_0$ as in Proposition 2 and 4. From (21) and by applying (F4), (18) becomes

$$\frac{\alpha}{2} - m(\nu||\hat{x} - \hat{y}||^2) < 0 \tag{22}$$

We let $\delta \to 0$ and divide the situation into two cases depending on the behavior of η as $\delta \to 0$.

Case 1. $\eta = \hat{x} - \hat{y} \to 0$ as $\delta \to 0$. First, we observe that $\nu \to 0$. Then by applying limits in (22) we obtain

$$\frac{\alpha}{2} < 0$$

which contradicts $\alpha > 0$.

Case 2. $\eta = \hat{x} - \hat{y} \to a \neq 0$ for some subsequence $\delta_j \to 0$. If we take $\lambda = \frac{\epsilon}{||\eta||^2}$, we have

$$\nu||\hat{x} - \hat{y}||^2 = \frac{21||\hat{x} - \hat{y}||^4}{\epsilon} + 12\delta||\hat{x} - \hat{y}||^2.$$

Letting $\delta_j \to 0$ in (22), it follows

$$\frac{\alpha}{2} - m(21\frac{||a||^4}{\epsilon}) < 0. \tag{23}$$

By Proposition 5, after letting $\gamma \to 0$, we see

$$\frac{||a||^4}{\epsilon} \to 0 \quad \text{as} \quad \epsilon \to 0.$$

Then, from (23), we obtain $\frac{\alpha}{2} < 0$, which is a contradiction. $\square$

References

[1] G. Barles, S. Biton, M. Bourgoing and O. Ley, *Quasilinear parabolic equations, unbounded solutions and geometrical equations III. Uniqueness through classical viscosity solutions methods*,

[2] M. G. Crandall and H. Ishii *The maximum principle for semicontinuous function*, Differential and Integral Equations 3 (1990), 1001-1014.

[3] Y. Giga, S. Goto, H. Ishii and M. H. Sato, *Comparison Principle and Convexity Preserving Properties for Singular Degenerate Parabolic Equations on Unbounded Domains*, Indiana Univ. Math. Journal, 40(2), 443-470, 1991.