THE EXTENSION OF A MONOTONE SET OPERATOR TO AN UPPER SEMICONTINUOUS SET OPERATOR

The purpose is to present a simple way to extend a translation-invariant monotone set operator $\mathcal{T}$ to an operator $\overline{\mathcal{T}}$ that is upper semicontinuous (u.s.c.) on the compact subsets of $\mathbb{T}^N$. Here are the basic assumptions and notation:

(1) $\mathcal{T}$ is monotone and translation invariant.
(2) $\mathcal{T}\emptyset = \emptyset$, $\mathcal{T}\mathbb{T}^N = \mathbb{T}^N$.
(3) $\mathcal{L}$ denotes the family of compact subsets of $\mathbb{T}^N$.
(4) The set of structuring elements of $\mathcal{T}$ relative to $\mathcal{L}$ is denoted by $\mathbb{B}$ and defined by $\mathbb{B} = \{X \mid 0 \in \mathcal{T}X, X \in \mathcal{L}\}$.

Definition 1. For $X \in \mathcal{L}$,

$$\overline{\mathcal{T}}X = \bigcap_{\mu<0} \mathcal{T}\mathcal{X}_\mu(-d(\cdot\,, X)),$$

where $d(\cdot\,, X)$ is the distance function of X.

1. General Results

The first result states that $\overline{\mathcal{T}}$ is indeed an extension of $\mathcal{T}$.

Result 1. $\mathcal{T}X \subset \overline{\mathcal{T}}X$.

Proof. $X = \mathcal{X}_0(-d(\cdot\,, X)) = \bigcap_{\mu<0} \mathcal{X}_\mu(-d(\cdot\,, X))$, and it is always true that

$$\mathcal{T}X = \mathcal{T}\Big(\bigcap_{\mu<0} \mathcal{X}_\mu(-d(\cdot\,, X))\Big) \subset \bigcap_{\mu<0} \mathcal{T}\mathcal{X}_\mu(-d(\cdot\,, X)) = \overline{\mathcal{T}}X. \qquad \square$$

Result 2. *Let $\overline{\mathbb{B}}$ denote all compact sets of the form $\overline{B} = \bigcap_n B_n$, where (B_n) is a descending sequence in $\mathbb{B}$. Then $\overline{\mathbb{B}}$ is the structuring set for $\overline{\mathcal{T}}$ relative to $\mathcal{L}$.*

Proof. Let $\mathbb{B}' = \{X \mid 0 \in \overline{\mathcal{T}}X, X \in \mathcal{L}\}$. If

$$0 \in \overline{\mathcal{T}}X = \bigcap_{\mu<0} \mathcal{T}\mathcal{X}_\mu(-d(\cdot\,, X)),$$

then $0 \in \mathcal{T}\mathcal{X}_\mu(-d(\cdot\,, X))$ for all $\mu < 0$. This means that all of the sets $\mathcal{X}_\mu(-d(\cdot\,, X))$ are in $\mathbb{B}$, and so $X \in \overline{\mathbb{B}}$. Hence, $\mathbb{B}' \subset \overline{\mathbb{B}}$.

Now assume that $\overline{B} \in \overline{\mathbb{B}}$ and that $B_n \downarrow \overline{B}$ for some descending sequence (B_n) in $\mathbb{B}$. By definition

$$\overline{\mathcal{T}}\,\overline{B} = \bigcap_{\mu<0} \mathcal{T}\mathcal{X}_\mu(-d(\cdot\,, \overline{B})).$$

Since d is continuous, $\overline{B} \subset (\mathcal{X}_\mu(-d(\cdot\,, \overline{B})))^\circ$ for all $\mu < 0$, where X° denotes the interior of X. Since $B_n \downarrow \overline{B}$, this implies that for each $\mu < 0$,

$$B_n \subset (\mathcal{X}_\mu(-d(\cdot\,, \overline{B})))^\circ \subset \mathcal{X}_\mu(-d(\cdot\,, \overline{B}))$$

for all sufficiently large n. As a consequence, $0 \in \mathcal{T}\mathcal{X}_\mu(-d(\cdot\,, \overline{B}))$ for all $\mu < 0$. Thus, $0 \in \overline{\mathcal{T}}\,\overline{B}$, and $\overline{B} \in \mathbb{B}'$. $\square$

The proof used this fact: If compact sets B_n are nonempty and $B_n \downarrow B$, then given any neighborhood O of B, $B_n \subset O$ for all sufficiently large n. This property of descending sequences of nonempty compact sets is at the heart of many of the following arguments.

The next result plays a central role in our development. It says that $\overline{\mathbb{B}}$, which was defined in terms of descending sequences in $\mathbb{B}$, is closed under descending sequences. In other words, taking descending sequences in $\overline{\mathbb{B}}$ adds no sets that were not already obtained by taking descending sequences in $\mathbb{B}$. It is this result that allows us to avoid using the Hausdorff metric directly.

Result 3. *If $C_n \downarrow C$, where $C_n \in \overline{\mathbb{B}}$, then there is a sequence $B_n \in \mathbb{B}$ such that $B_n \downarrow C$.*

Proof. Assume that $C_n \downarrow C$, where $C_n \in \overline{\mathbb{B}}$. For each n, by the definition of $\overline{\mathbb{B}}$, we can find a set $A_n \in \mathbb{B}$ such that $C_n \subset A_n$ and $A_n \subset D_{1/n}(C_n)$, where $D_\varepsilon(X) = \{\boldsymbol{y} \mid |\boldsymbol{x} - \boldsymbol{y}| \leq \varepsilon \text{ for some } \boldsymbol{x} \in X\}$. Now define $E_n = \overline{\bigcup_{k \geq 0} A_{n+k}}$. The E_n form a descending sequence in $\mathbb{B}$. Let $E = \bigcap_n E_n$. We wish to show that $E = C$. Since $C_n \subset A_n$ for all n, and since $A_n \subset E_n$, $C \subset E$.

Since $C_n \downarrow C$, given any $\varepsilon > 0$, there is an $N = N(\varepsilon)$ such that $C_n \subset D_\varepsilon(C)$ for all $n \geq N$. This implies that

$$A_n \subset D_{1/n}(C_n) \subset D_{(1/n)+\varepsilon}(C)$$

for all $n \geq N$. If $m > n$, $A_m \subset D_{(1/m)+\varepsilon}(C) \subset D_{(1/n)+\varepsilon}(C)$. This implies that $E_n \subset D_{(1/n)+\varepsilon}(C)$ for all $n \geq N$, which in turn implies that $E \subset D_\varepsilon(C)$. Since ε was arbitrary, we see that $E \subset C$. □

Result 4. *$\overline{\mathcal{T}}$ is upper semicontinuous on $\mathcal{L}$.*

Proof. Assume that X_n is a descending sequence of nonempty compact sets in $\mathbb{T}^N$ and that $X = \bigcap_n X_n$. Assume that $\boldsymbol{x} \in \bigcap_n \overline{\mathcal{T}} X_n$. We must show that there is a $\overline{B} \in \overline{\mathbb{B}}$ such that $\boldsymbol{x} + \overline{B} \subset X$.

The assumption that $\boldsymbol{x} \in \bigcap_n \overline{\mathcal{T}} X_n$ implies that for each n there is a set $A_n \in \overline{\mathbb{B}}$ such that $\boldsymbol{x} + A_n \subset X_n$. Since the sequence X_n is descending, $m > n$ implies that $\boldsymbol{x} + A_m \subset X_n$. Thus, $\boldsymbol{x} + \overline{\bigcup_{k \geq 0} A_{n+k}} \subset X_n$. Since both sides of this inclusion are descending, it follows that $\boldsymbol{x} + \bigcap_n \overline{\bigcup_{k \geq 0} A_{n+k}} \subset X$. We showed in the Result 3 that $\bigcap_n \overline{\bigcup_{k \geq 0} A_{n+k}}$ was in $\overline{\mathbb{B}}$, so this proves that $\overline{\mathcal{T}}$ is upper semicontinuous. □

So far, we have only considered the operators $\mathcal{T}$ and $\overline{\mathcal{T}}$ to be defined on compact sets, but, in fact, nothing prevents extending their domain of definition to all subsets of $\mathbb{T}^N$ by the rule that says $\boldsymbol{x} \in \overline{\mathcal{T}} X$ if and only if there is a $\overline{B} \in \overline{\mathbb{B}}$ such that $\boldsymbol{x} + \overline{B} \subset X$. The next two results say something more about how the operators $\mathcal{T}$ and $\overline{\mathcal{T}}$ are related.

Result 5. *If X is open, then $\overline{\mathcal{T}} X = \mathcal{T} X$ and $\overline{\mathcal{T}} X$ is open.*

Proof. Assume that X is open. By definition, $\boldsymbol{x} \in \overline{\mathcal{T}} X$ if and only if there is a $\overline{B} \in \overline{\mathbb{B}}$ such that $\boldsymbol{x} + \overline{B} \subset X$. This means that $X - \boldsymbol{x}$ is a neighborhood of the compact set $\overline{B}$. Thus, if $B_n \downarrow \overline{B}$, then $B_n \subset X - \boldsymbol{x}$ for all sufficiently large n. This implies that $\boldsymbol{x} \in \mathcal{T} X$. Since $\mathcal{T} X \subset \overline{\mathcal{T}} X$, this proves that $\overline{\mathcal{T}} X = \mathcal{T} X$.

Now suppose that $\boldsymbol{x} + B \subset X$, that is, $\boldsymbol{x} \in \mathcal{T} X$. Then the compact set B is in the open set $X - \boldsymbol{x}$. This implies that the distance between $\mathbb{T}^N \setminus X - \boldsymbol{x}$ and B is positive. Thus, there is an $\varepsilon > 0$ such that $B + \boldsymbol{y} \in X - \boldsymbol{x}$ for all $|\boldsymbol{y}| < \varepsilon$. Written another way, this says that $\boldsymbol{x} + \boldsymbol{y} + B \subset X$ for all $|\boldsymbol{y}| < \varepsilon$, which in turn means that $\boldsymbol{x} + \boldsymbol{y} \in \mathcal{T} X$ for all $|\boldsymbol{y}| < \varepsilon$. Hence, $\mathcal{T} X$ is open. □

Result 6. *If X is compact, then $\overline{\mathcal{T}} X$ is compact.*

Proof. Assume that $\boldsymbol{x}_n \to \boldsymbol{x}$, $\boldsymbol{x}_n \in \overline{\mathcal{T}}X$. We must show that $\boldsymbol{x} \in \overline{\mathcal{T}}X$. Since $\boldsymbol{x}_n \in \overline{\mathcal{T}}X$, for each n there is a set $A_n \in \overline{\mathbb{B}}$ such that $\boldsymbol{x}_n + A_n \subset X$. If we write this as $A_n \subset X - \boldsymbol{x}_n$, then $\bigcup_{k\geq 0} A_{n+k} \subset \bigcup_{k\geq 0}(X - \boldsymbol{x}_{n+k})$. Given $\varepsilon > 0$, there is an $N = N(\varepsilon)$ such that $|\boldsymbol{x} - \boldsymbol{x}_n| \leq \varepsilon$ for all $n \geq N$. This implies that $\bigcup_{k\geq 0}(X - \boldsymbol{x}_{n+k}) \subset \bigcup_{|\boldsymbol{x}-\boldsymbol{y}|\leq\varepsilon}(X - \boldsymbol{y})$ for all $n \geq N$. The set on the right-hand of this inclusion is $D_\varepsilon(X) - \boldsymbol{x}$. Since $\bigcup_{k\geq 0} A_{n+k} \subset \bigcup_{k\geq 0}(X - \boldsymbol{x}_{n+k})$, this implies that $\bigcap_n \overline{\bigcup_{k\geq 0} A_{n+k}} \subset D_\varepsilon(X) - \boldsymbol{x}$, or that $\boldsymbol{x} + \bigcap_n \overline{\bigcup_{k\geq 0} A_{n+k}} \subset D_\varepsilon(X)$. Since ε was arbitrary, $\boldsymbol{x} + \cap_n \overline{\bigcup_{k\geq 0} A_{n+k}} \subset X$. The set $\bigcap_n \overline{\bigcup_{k\geq 0} A_{n+k}}$ is in $\overline{\mathbb{B}}$ by Result 2. Thus, $\boldsymbol{x} \in \overline{\mathcal{T}}X$, which proves that $\overline{\mathcal{T}}X$ is compact. □

Result 7. *$\mathcal{T}$ is u.s.c. on $\mathcal{L}$ if and only if $\mathbb{B}$ is closed under descending sequences.*

Proof. The proof of Result 4 shows that $\mathcal{T}$ is u.s.c. on $\mathcal{L}$ if $\mathbb{B}$ is closed under descending sequences. Assume that $B = \bigcap B_n$, where $B_n \in \mathbb{B}$. If $\mathcal{T}$ is u.s.c. on $\mathcal{L}$, then $\mathcal{T}B = \bigcap_n \mathcal{T}B_n$. Since $0 \in \mathcal{T}B_n$ for all n, $0 \in \mathcal{T}B$, which proves that $B \in \mathbb{B}$. □

We are now going to shift attention from the set operators to their associated function operators. Thus, for $u : \mathbb{T}^N \to \mathbb{R}$, we define T and $\overline{T}$ by

$$Tu(\boldsymbol{x}) = \sup\{\lambda \mid \boldsymbol{x} \in \mathcal{T}\mathcal{X}_\lambda u\} \quad \text{and} \quad \overline{T}u(\boldsymbol{x}) = \sup\{\lambda \mid \boldsymbol{x} \in \overline{\mathcal{T}}\mathcal{X}_\lambda u\}.$$

Result 8. *If u is continuous, then $Tu = \overline{T}u$.*

Proof. Without loss of generality, we may assume that u maps $\mathbb{T}^N$ onto $[0, 1]$. With this assumption, $\mathcal{X}_\lambda u = \emptyset$ for $\lambda > 1$, $\mathcal{X}_\lambda u = \mathbb{T}^N$ for $\lambda \leq 0$, and $\mathcal{X}_\nu u \subset (\mathcal{X}_\mu u)^\circ$ for $\mu < \nu \leq 1$, where X° denotes the interior of X. This last relation implies that $\bigcap_{\mu<\lambda} \mathcal{T}(\mathcal{X}_\mu u)^\circ = \bigcap_{\mu<\lambda} \mathcal{T}\mathcal{X}_\mu u$ and that $\bigcap_{\mu<\lambda} \overline{\mathcal{T}}(\mathcal{X}_\mu u)^\circ = \bigcap_{\mu<\lambda} \overline{\mathcal{T}}\mathcal{X}_\mu u$. Since $\mathcal{T}$ and $\overline{\mathcal{T}}$ agree on open sets, $\bigcap_{\mu<\lambda} \mathcal{T}\mathcal{X}_\mu u = \bigcap_{\mu<\lambda} \overline{\mathcal{T}}\mathcal{X}_\mu u$. This implies that Tu and $\overline{T}u$ have the same level sets: $\mathcal{X}_\lambda Tu = \bigcap_{\mu<\lambda} \mathcal{T}\mathcal{X}_\mu u = \bigcap_{\mu<\lambda} \overline{\mathcal{T}}\mathcal{X}_\mu u = \mathcal{X}_\lambda \overline{T}u$. Since they have the same level sets, $Tu = \overline{T}u$ for u continuous. □

Result 9. *If u is upper semicontinuous, then $\overline{T}u$ is upper semicontinuous.*

Proof. By the definition of upper semicountinuity, $\mathcal{X}_\lambda u$ is closed and hence compact. Then by Result 6, $\overline{\mathcal{T}}\mathcal{X}_\lambda u$ is compact. Hence, $\bigcap_{\mu<\lambda} \overline{\mathcal{T}}\mathcal{X}_\mu u = \mathcal{X}_\lambda \overline{T}u$ is compact, which means that $\overline{T}u$ is upper semicontinuous. □

Result 10. *If u is continuous, then $\overline{T}u$ is continuous.*

Proof. Without loss of generality, we assume that u maps $\mathbb{T}^N$ onto $[0, 1]$. Result 9 states that u is upper semicontinuous. Thus, to show that $\overline{T}u$ is continuous, it is sufficient to show that $\{\boldsymbol{x} \mid \overline{T}u(\boldsymbol{x}) > \lambda\} = \bigcup_{\mu>\lambda} \mathcal{X}_\mu \overline{T}u$ is open. The fact that $\overline{\mathcal{T}}$ is u.s.c. on compact sets implies that $\mathcal{X}_\mu \overline{T}u = \overline{\mathcal{T}}\mathcal{X}_\mu u$ for u continuous. This reduces the task to showing that $\bigcup_{\mu>\lambda} \overline{\mathcal{T}}\mathcal{X}_\mu u$ is open.

Since u is continuous, we know that $\mathcal{X}_\mu u \subset (\mathcal{X}_\nu u)^\circ \subset \mathcal{X}_\nu u$ for $\nu < \mu \leq 1$, and hence that $\overline{\mathcal{T}}\mathcal{X}_\mu u \subset \overline{\mathcal{T}}(\mathcal{X}_\nu u)^\circ \subset \overline{\mathcal{T}}\mathcal{X}_\nu u$. If $\boldsymbol{x} \in \bigcup_{\mu>\lambda} \overline{\mathcal{T}}\mathcal{X}_\mu u$, then $\boldsymbol{x} \in \overline{\mathcal{T}}\mathcal{X}_\mu u$ for some μ, where $\lambda < \mu \leq 1$. Then for any ν, $\lambda < \nu < \mu$, we have the inclusions

$$\boldsymbol{x} \in \overline{\mathcal{T}}\mathcal{X}_\mu u \subset \overline{\mathcal{T}}(\mathcal{X}_\nu u)^\circ \subset \overline{\mathcal{T}}\mathcal{X}_\nu u.$$

The set $\overline{\mathcal{T}}(\mathcal{X}_\nu u)^\circ$ is open, hence, $\overline{\mathcal{T}}(\mathcal{X}_\nu u)^\circ \subset (\overline{\mathcal{T}}\mathcal{X}_\nu u)^\circ$, and this means that $\boldsymbol{x}$ is an interior point of $\overline{\mathcal{T}}\mathcal{X}_\nu u$. The conclusion is that $\bigcup_{\mu>\lambda} \overline{\mathcal{T}}\mathcal{X}_\mu u$ is open, and hence that $\overline{T}u$ is continuous. □

Result 11. *The operator $\overline{T}$ is the Evans–Spruck extension of the operator T.*

Proof. Let T' denote the Evans–Spruck extension of T. Since both $\overline{\mathcal{T}}$ and $\mathcal{T}'$ are u.s.c. on $\mathcal{L}$, it is always true that $\overline{\mathcal{T}}\mathcal{X}_\lambda u = \mathcal{X}_\lambda \overline{T} u$ and $\mathcal{T}'\mathcal{X}_\lambda u = \mathcal{X}_\lambda T' u$ whenever u is continuous. Since the three operators $\overline{\mathcal{T}}$, T', and T agree on continuous functions, we see that $\overline{\mathcal{T}}\mathcal{X}_\lambda u = \mathcal{T}'\mathcal{X}_\lambda u$ for u continuous. This, in turn, implies that $\overline{\mathcal{T}}X = \mathcal{T}'X$ for any $X \in \mathcal{L}$. □

Result 12. *If $\mathcal{T}'$ is an u.s.c. extension of $\mathcal{T}$ and if $Tu = T'u$ for u continuous, then $\mathcal{T}' = \overline{\mathcal{T}}$.*

Proof. The proof is exactly the same as the proof of the last result. □

Result 13. *If $\mathcal{T}'$ is an u.c.s. extension of $\mathcal{T}$ and if $\mathcal{T}X \subset \mathcal{T}'X \subset \overline{\mathcal{T}}X$, then $\mathcal{T}'X = \overline{\mathcal{T}}X$ for X compact.*

Proof. The inclusions $\mathcal{T}X \subset \mathcal{T}'X \subset \overline{\mathcal{T}}X$ imply that $\mathbb{B} \subset \mathbb{B}' \subset \overline{\mathbb{B}}$. Result 7 states that $\mathbb{B}'$ is closed under descending sequences. This implies by Result 3 that $\overline{\mathbb{B}} \subset \mathbb{B}'$. Hence, $\mathbb{B}' = \overline{\mathbb{B}}$, and $\mathcal{T}'$ and $\overline{\mathcal{T}}$ agree on compact set. □

2. Examples and Applications

Example 1. Let $\mathcal{T}^\circ$ denote the set operator defined by $\mathcal{T}^\circ X = X^\circ$, and restrict the domain of definition of $\mathcal{T}^\circ$ to $\mathcal{L}$. The family of structuring elements for $\mathcal{T}^\circ$, $\mathbb{B} = \{X \mid 0 \in \mathcal{T}^\circ X = X^\circ, X \in \mathcal{L}\}$, is the family of all compact sets X such that $0 \in X^\circ$. Clearly, $\{0\} \in \overline{\mathbb{B}}$, and this means that $\overline{\mathcal{T}^\circ} = \mathcal{I}$, the identity set operator.

Example 2. Let $\mathcal{T}_a$ denote the small component killer restricted to $\mathcal{L}$. By definition,

$$\mathcal{T}_a X = \bigcup_{|X_i| \geq a} X_i,$$

where the X_i are the connected components of X and $|X|$ denotes the measure of X. Then the family of structuring elements for $\mathcal{T}_a$ can be characterized as follows: $\mathbb{B}$ consists of the compact sets X that have a finite number of connected components with measure greater then or equal to a. If $B_n \downarrow B$ and the B_n are connected compact sets with measure greater than or equal to a, then B is a connected set and measure$(B) \geq a$. It follows that $\mathbb{B}$ is closed under descending sequences, and hence that $\mathcal{T}_a$ is u.s.c.

Example 3. Let $\mathcal{E}$ be the erosion operator defined by the set B, that is,

$$\mathcal{E}X = \{\boldsymbol{x} \mid \boldsymbol{x} + B \subset X\}.$$

Then the structuring elements $\mathbb{B}$ for $\mathcal{E}$ restricted to the compact sets is just the set of all compacts sets that contain B. Clearly, the closure of B, denoted by $\overline{B}$, is in $\mathbb{B}$, and $\mathbb{B} = \overline{\mathbb{B}}$. Thus, $\mathcal{E}$ restricted to $\mathcal{L}$ is u.s.c. on $\mathcal{L}$. In this case, $\overline{\mathbb{B}}$ contains a minimal element, and $\mathcal{E}$ restricted to $\mathcal{L}$ is generated by $\overline{B}$.

Example 4. Let $\mathcal{D}$ be the dilation operator defined by the set B, that is,

$$\mathcal{D}X = \{\boldsymbol{y} \mid \boldsymbol{y} = \boldsymbol{x} + \boldsymbol{b}, \boldsymbol{x} \in X, \boldsymbol{b} \in B\}.$$

Then the structuring elements for $\mathcal{D}$ restricted to $\mathcal{L}$ is the family of sets $\mathbb{B}$ defined by

$$\mathbb{B} = \{X \mid X \in \mathcal{L}, -\boldsymbol{b} \in X \text{ for some } \boldsymbol{b} \in B\}.$$

In this case, $\mathbb{B}$ is not necessarily closed under descending sequences, but it is easy to see that

$$\overline{\mathbb{B}} = \{X \mid X \in \mathcal{L}, -\boldsymbol{b} \in X \text{ for some } \boldsymbol{b} \in \overline{B}\}.$$

The family $\mathbb{B}^* = \{\{\boldsymbol{b}\} \mid \boldsymbol{b} \in \overline{B}\}$ also generates the extension $\overline{\mathcal{D}}$, and it is minimal in the sense that if $\mathbb{B}'$ generates $\overline{\mathcal{D}}$ then $\mathbb{B}^* \subset \mathbb{B}'$.

Example 5. Let $\mathcal{M}_k$ denote the median filter based on the weight function k. Then $\mathcal{M}_k$ restricted to $\mathcal{L}$ is defined by

$$\mathcal{M}_k X = \{\boldsymbol{x} \mid |X - \boldsymbol{x}|_k \geq 1/2\}.$$

Then $\mathbb{B} = \{B \mid B \in \mathcal{L}, |B|_k \geq 1/2\}$. Thanks to M. Lebesgue, this family is clearly closed under descending sequences, and thus $\mathcal{M}_k$ is u.s.c. on $\mathcal{L}$.

3. A Final Question

As the examples show, a set operator can be defined directly in terms of its structuring elements as in Example 3, or it can be defined in other terms, in which case the structuring elements must be determined. Suppose that $\mathcal{T}$ is defined in terms of a family of sets $\mathbb{B}$ without specifying any specific domain for $\mathcal{T}$. The question that arises is, How are the elements of $\mathbb{B}$ related to the elements of the structuring set of $\mathcal{T}$ restricted to $\mathcal{L}$? Let $\mathbb{B}_{\mathcal{T}}$ denote the structuring set of $\mathcal{T}$ restricted to $\mathcal{L}$. Then $\mathbb{B}_{\mathcal{T}} = \{X \mid X \in \mathcal{L}, B \subset X\}$. There is a "smaller" family $\mathbb{B}^* = \{\overline{B} \mid B \in \mathbb{B}\}$ that also generates $\mathcal{T}$ restricted to $\mathcal{L}$. We saw an instance of this in Example 3.

4. Comments

I believe that this development can also be done in the following context:

(a) The underlying space is $\mathbb{R}^N$.

(b) The only functions that appear are continuous whose level sets are compact.

(c) $\mathcal{T}$ is defined on the compact subsets of $\mathbb{R}^N$, $\mathcal{T}\emptyset = \emptyset$, and $\mathcal{T}\mathbb{R}^N = \mathbb{R}^N$.

The problem I have with $\mathbb{T}^N$ comes later when we wish to consider Jordan curves. There are at least two kinds of Jordan curves on $\mathbb{T}^2$, and the ones we wish to consider are that ones that can be continuously deformed to a point. I guess all is OK if one only considers curves that are inside the fundamental square in $\mathbb{R}^2$.

Result 12 says that an u.s.c. extension $\mathcal{T}'$ of $\mathcal{T}$ is unique if we require that T and T' agree on continuous functions. Result 13 says that $\overline{\mathcal{T}}$ is the minimal u.s.c. extension of $\mathcal{T}$ satisfying the condition $\mathcal{T}X \subset \mathcal{T}'X$ for $X \in \mathcal{L}$. There is, however, not necessarily a maximal extension for which $\mathcal{T}X \subset \mathcal{T}'X$ for $X \in \mathcal{L}$. For example, by adding more sets of the form $\{\boldsymbol{c}\}$ to the family $\mathbb{B}^*$ in Example 4, you always get a proper u.s.c. extension of $\overline{\mathcal{T}}$.