Subdivision methods #1 - Bezier curves

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1 Points and curves

A Bezier curve $\mathcal{C}_0^0$ can be defined from its three control points

$$\mathcal{P}^0 = \{P_0^0, P_1^0, P_2^0\}$$

by the parametric function

$$\mathcal{C}_0^0 = \{(1-t)^2 P_0^0 + 2t(1-t)P_1^0 + t^2 P_2^0, t \in [0;1[\}$$

An interesting property of this curve is that if we get five points

$$\mathcal{P}^1 = P_0^1, P_1^1, P_2^1, P_3^1, P_4^1$$

from $\mathcal{P}^0$ with a simple transformation, and then define two new Bezier curves $\mathcal{C}_0^1$ (*resp.* $\mathcal{C}_1^1$) from the triplets $\mathcal{T}_0^1 = \{P_0^1, P_1^1, P_2^1\}$ (*resp.* $\mathcal{T}_1^1 = \{P_2^1, P_3^1, P_4^1\}$), then each of these two little Bezier curves is half of the original one:

$$\mathcal{C}_0^0 = \mathcal{C}_0^1 \cup \mathcal{C}_1^1$$

$\mathcal{C}_0^1$ and $\mathcal{C}_1^1$ are both Bezier curves, so the transformation can continue to a next step with nine points and four Bezier curves, and eight curves for next step, and so on; from the relation between $\mathcal{P}_0$ and $\mathcal{P}_1$, we will demonstrate the existence of this subdivision process.

Let's first detail the points and triplets generating process, and introduce the notation.

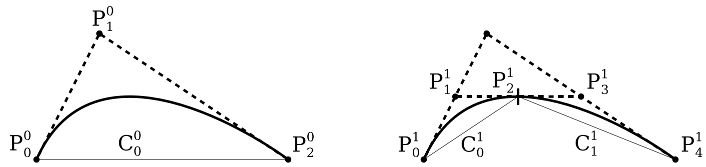

Figure 1: The Bezier curves subdivision process, from three points and one curve to five points and two curves.

step 0 The initial set, $\mathcal{P}^0$, contains one triplet $\mathcal{T}_0^0 = \{P_0^0, P_1^0, P_2^0\}$; this triplet defines the Bezier curve $\mathcal{C}_0^0$, as explained before:

$$\mathcal{C}_0^0 = \{M_0^0(t), t \in [0;1[\}, \text{ with } M_0^0(t) = b_0^0(t)P_0^0 + b_1^0(t)P_1^0 + b_2^0(t)P_2^0$$

where b_0, b_1 and b_2 are the Bernstein functions, defined for $t \in [0;1[$ and null out of this interval:

$$\begin{aligned} b_0^0(t) &= (1-t)^2 \\ b_1^0(t) &= 2t(1-t) \\ b_2^0(t) &= t^2 \end{aligned}$$

We can remark that $M_0^0(0) = P_0^0$ and $\lim_{t\to 1} M_0^0(t) = P_2^0$: $\mathcal{C}_0^0$ starts on P_0^0 and ends on P_2^0.

step 1 The first step set, $\mathcal{P}^1$, contains five points $P_0^1, P_1^1, P_2^1, P_3^1, P_4^1$, defined from $\mathcal{P}^0$ by the following formulas

$$\begin{aligned} P_0^1 &= P_0^0 \\ P_1^1 &= \tfrac{1}{2}(P_0^0 + P_1^0) \\ P_2^1 &= \tfrac{1}{2}(P_1^1 + P_3^1) = \tfrac{1}{4}P_0^0 + \tfrac{1}{2}P_1^0 + \tfrac{1}{4}P_2^0 \\ P_3^1 &= \tfrac{1}{2}(P_1^0 + P_2^0) \\ P_4^1 &= P_2^0 \end{aligned}$$

Two triplets $\mathcal{T}_0^1 = \{P_0^1, P_1^1, P_2^1\}$ and $\mathcal{T}_1^1 = \{P_2^1, P_3^1, P_4^1\}$ are defined in $\mathcal{P}^1$, such that $\mathcal{T}_0^1 \cup \mathcal{T}_1^1 = \mathcal{P}^1$ and $\mathcal{T}_0^1 \cap \mathcal{T}_1^1 = P_2^1$ (the last point of $\mathcal{T}_0^1$ is the first point of $\cap\mathcal{T}_1^1$).

Looking at M_0^0 defined before, we remark that $M_0^0(0) = P_0^1$, $M_0^0(1/2) = P_2^1$ and $\lim_{t\to 1} M_0^0(t) = P_4^1$: $\mathcal{C}_0^0$ starts on P_0^1, is on P_2^1 at middle-course, and ends on P_4^1.

We consider now the curves C_0^1 and C_1^1, defined from $\mathcal{T}_0^1$ and $\mathcal{T}_1^1$ with the same formulas as C_0^0:

$$\mathcal{C}_0^1 = \{M_0^1(t), t \in [0;1[\}, \text{ with } M_0^1(t) = b_0^0(t)P_0^1 + b_1^0(t)P_1^1 + b_2^0(t)P_2^1$$

$$\mathcal{C}_1^1 = \{M_1^1(t), t \in [0;1[\}, \text{ with } M_1^1(t) = b_0^0(t)P_2^1 + b_1^0(t)P_3^1 + b_2^0(t)P_4^1$$

These curves are Bezier curves, because they are defined by a triplet and the associated parametric function; we also remark that $\mathcal{C}_0^1$ starts on P_0^1 and ends on P_2^1 while $\mathcal{C}_1^1$ starts on P_2^1 and ends on P_4^1.

Then, $\forall t \in [0;1[$,

$$\begin{aligned} M_0^1(t) &= b_0^0(t)P_0^1 + b_1^0(t)P_1^1 + b_2^0(t)P_2^1 \\ &= b_0^0(t)P_0^0 + b_1^0(t)\tfrac{1}{2}(P_0^0 + P_1^0) + b_2^0(t)(\tfrac{1}{4}P_0^0 + \tfrac{1}{2}P_1^0 + \tfrac{1}{4}P_2^0) \\ &= (b_0^0(t) + \tfrac{1}{2}b_1^0(t) + \tfrac{1}{4}b_2^0(t))P_0^0 + \tfrac{1}{2}(b_1^0(t) + b_2^0(t))P_1^0 + \tfrac{1}{4}b_2^0(t)P_2^0 \\ &= ((1-t)^2 + t(1-t) + \tfrac{1}{4}t^2)P_0^0 + (t(1-t) + \tfrac{1}{2}t^2)P_1^0 + \tfrac{1}{4}t^2P_2^0 \\ &= (1-t/2)^2P_0^0 + t(1-t/2)P_1^0 + (t/2)^2P_2^0 \\ &= M_0^0(t/2) \end{aligned}$$

And $\forall t \in [0;1[, M_1^1(t) = M_0^0(t/2+1/2)$, with the same equations. We can also write

$$\begin{array}{ll} M_0^0(t) = M_0^1(2t) & \forall t \in [0;1/2[\\ M_0^0(t) = M_1^1(2t-1) & \forall t \in [1/2;1[\end{array}$$

This shows that the Bezier curve defined on the first (*resp. second*) triplet obtained from the initial triplet is the first (*resp. second*) half of the Bezier curve defined n the initial triplet; this matches the previously explained position of the parametric curves, for $t = 0, 1/2, 1$.

All these properties are recursive; we will expand them to the i^{th} iteration.

step i The i^{th} set $\mathcal{P}^i$ contains $2^{i+1}+1$ points $P_0^i, P_1^i, ..., P_{2^{i+1}-1}^i, P_{2^{i+1}}^i$, defined from the ones in $\mathcal{P}^{i-1}$ by a recursive relation:

$$\forall i > 0, \forall k \in \{0..2^{i+1}\},$$

$$P_k^i = \begin{cases} P_{k/2}^{i-1} & \text{if } k \bmod 4 = 0 \\ \frac{1}{2}(P_{(k-1)/2}^{i-1} + P_{(k+1)/2}^{i-1}) & \text{if } k \bmod 4 = 1 \text{ or } 3 \\ \frac{1}{4}P_{(k-2)/2}^{i-1} + \frac{1}{2}P_{k/2}^{i-1}) + \frac{1}{4}P_{(k+2)/2}^{i-1}) & \text{if } k \bmod 4 = 2 \end{cases}$$

We can define 2^i triplets $\mathcal{T}_j^i, j = 0..2^i - 1$ in $\mathcal{P}^i$, with

$$\mathcal{T}_j^i = \{P_2^i j, P_{2j+1}^i, P_{2j+2}^i\}$$

such that the last point of $\mathcal{T}_j^i$ is the first point of $\mathcal{T}_{j+1}^i$. We can note that $\mathcal{T}_j^i$ can be defined using only one triplet, $\mathcal{T}_{j/2}^{i-1}$ or $\mathcal{T}_{(j-1)/2}^{i-1}$, from the previous step.

We also have $\mathcal{C}_j^i$, the Bezier curve defined on the triplet $\mathcal{T}_j^i$ from the usual formula:

$$\mathcal{C}_j^i = \{M_j^i(t), t \in [0;1[\}, \text{ with } M_j^i(t) = b_0^0(t)P_2^ij + b_1^0(t)P_{2j+1}^i + b_2^0(t)P_{2j+2}^i$$

The previous properties are transmitted :

$$\begin{array}{ll} \forall i,j, & M_j^i(0) = P_2^ij \\ & M_j^i(1/2) = P_{4j+2}^{i+1} \\ & \lim_{t\to 1} M_j^i(t) = P_{2j+2}^i = M_{j+1}^i(0) \end{array}$$

and $\forall t \in [0;1[$,

$$\begin{array}{lll} M_j^i(t) & = M_{j/2}^{i-1}(t/2) & \text{if } j \bmod 2 = 0 \\ & = M_{(j-1)/2}^{i-1}(t/2+1/2) & \text{if } j \bmod 2 = 1 \end{array}$$

and

$$\begin{array}{ll} M_j^i(t) = M_2^{i+1}j(2t) & \forall t \in [0;1/2[\\ M_j^i(t) = M_{2j+1}^{i+1}(2t-1) & \forall t \in [1/2;1[\end{array}$$

And we have a general expression of the Bezier curves substitution:

$$\forall i,j, \mathcal{C}_j^i = \mathcal{C}_2^{i+1}j \cup \mathcal{C}_{2j+1}^{i+1}$$

$$\forall i,j,m, \mathcal{C}_j^i = \bigcup_{n=0}^{2^m-1} \mathcal{C}_{2^mj+n}^{i+m}$$

2 Bernstein functions

From now we will only consider step 0 and step 1; the situation is exactly the same at step i and $i+1$, but restricting to 0 and 1 makes the notations more easy. We have the following parametric curves, defined with $V^0 = \{b_0^0, b_1^0, b_2^0\}$:

$$M_0^0(t) = b_0^0(t)P_0^0 + b_1^0(t)P_1^0 + b_2^0(t)P_2^0$$
$$M_0^1(t) = b_0^0(t)P_0^1 + b_1^0(t)P_1^1 + b_2^0(t)P_2^1$$
$$M_1^1(t) = b_0^0(t)P_2^1 + b_1^0(t)P_3^1 + b_2^0(t)P_4^1$$

Then, we introduce five parametric functions $V^1 = \{b_0^1, b_1^1, b_2^1, b_3^1, b_4^1\}$, such that

$$M_0^0(t) = b_0^1(t)P_0^1 + b_1^1(t)P_1^1 + b_2^1(t)P_2^1 + b_3^1(t)P_3^1 + b_4^1(t)P_4^1 \text{ for } t \in [0;1[$$

We can explicit these b_j^1 functions from the relations between the points, $\mathcal{P}^0$ and $\mathcal{P}^1$. The functions are null, except on the following intervals:

$$\begin{array}{ll}
b_0^1(t) = b_0^0(2t) = (1-2t)^2 & \text{for } t \in [0; 1/2[\\
b_1^1(t) = b_1^0(2t) = 4t(1-2t) & \text{for } t \in [0; 1/2[\\
b_2^1(t) = b_2^0(2t) = 4t^2 & \text{for } t \in [0; 1/2[
\end{array}$$

$$\begin{array}{ll}
b_2^1(t) = b_0^0(2t-1) = 4(t-1)^2 & \text{for } t \in [1/2; 1[\\
b_3^1(t) = b_1^0(2t-1) = 4(2t-1)(1-t) & \text{for } t \in [1/2; 1[\\
b_4^1(t) = b_2^0(2t-1) = (2t-1)^2 & \text{for } t \in [1/2; 1[
\end{array}$$

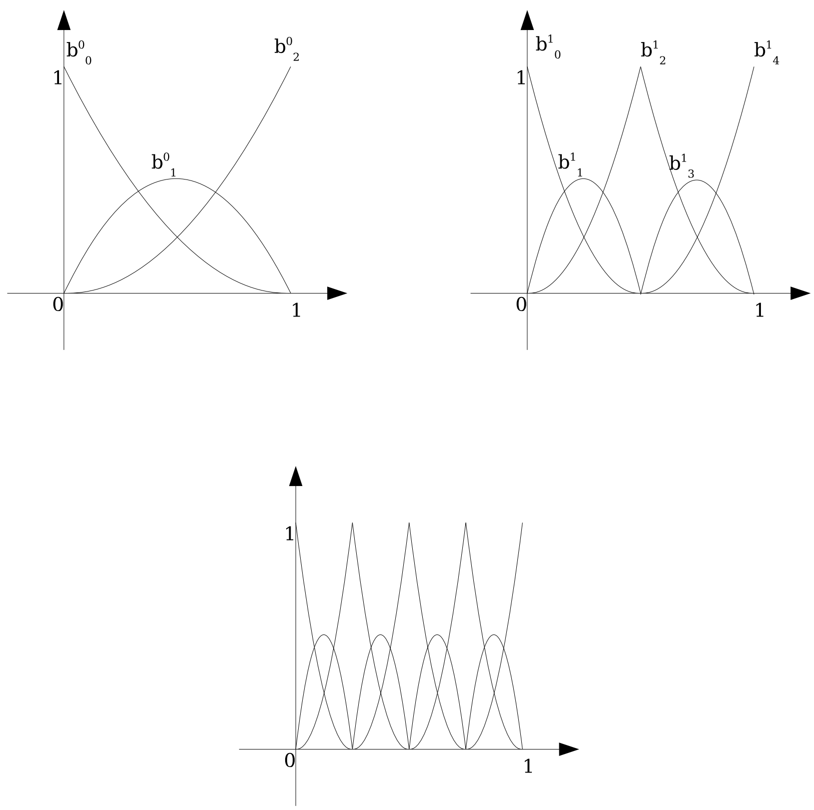

Figure 2: Three generations of the Bernstein functions for the Bezier curves.

This can be verified by developing the formulas: for $t \in [0;1/2[$,

$$\begin{aligned}
& b_0^1(t)P_0^1 + b_1^1(t)P_1^1 + b_2^1(t)P_2^1 \\
& = (1-2t)^2 P_0^1 + 4t(1-2t)P_1^1 + 4t^2 P_2^1 \\
& = (1-2t)^2 P_0^0 + 4t(1-2t)(\tfrac{1}{2}P_0^0 + \tfrac{1}{2}P_1^0) + 4t^2(\tfrac{1}{4}P_0^0 + \tfrac{1}{2}P_1^0 + \tfrac{1}{4}P_2^0) \\
& = (1-2t+t^2)P_0^0 + (2t-2t^2)P_1^0 + t^2 P_2^0 \\
& = b_0^0(t)P_0^0 + b_1^0(t)P_1^0 + b_1^1(t)P_1^1
\end{aligned}$$

and for $t \in [1/2;1[$,

$$\begin{aligned}
& b_2^1(t)P_2^1 + b_3^1(t)P_3^1 + b_4^1(t)P_4^1 \\
& = 4(t-1)^2 P_2^1 + 4(2t-1)(1-t)P_3^1 + (2t-1)^2 P_4^1 \\
& = 4(t-1)^2(\tfrac{1}{4}P_0^0 + \tfrac{1}{2}P_1^0 + \tfrac{1}{4}P_2^0) + 4(2t-1)(1-t)(\tfrac{1}{2}P_1^0 + \tfrac{1}{2}P_2^0) + (2t-1)^2 P_2^0 \\
& = (1-2t+t^2)P_0^0 + (2t-2t^2)P_1^0 + t^2 P_2^0 \\
& = b_0^0(t)P_0^0 + b_1^0(t)P_1^0 + b_1^1(t)P_1^1
\end{aligned}$$

So, for $t \in [0;1[$, the formulation with V^1 is equivalent to the formulation with V^0 :

$$b_0^1(t)P_0^1 + b_1^1(t)P_1^1 + b_2^1(t)P_2^1 + b_3^1(t)P_3^1 + b_4^1(t)P_4^1 = b_0^0(t)P_0^0 + b_1^0(t)P_1^0 + b_2^0(t)P_2^0$$

And graphically, we can easily understand that, from step 0 to step 1, we just have simple homothety and translations; this provides a method to obtain the b_j^i functions for the next subdivision steps, without extra calculations.

3 Functions spaces

We have now two sets of functions from $[0;1]$ to $[0;1]$, and the corresponding vector spaces

$$\begin{aligned}
V^0 &= \{b_0^0, b_1^0, b_2^0\}, \mathcal{V}^0 = vect(V^0) \\
V^1 &= \{b_0^1, b_1^1, b_2^1, b_3^1, b_4^1\}, \mathcal{V}^1 = vect(V^1)
\end{aligned}$$

The dimension of $\mathcal{V}^0$ is 3, and so B^0 is a basis, because

$$\alpha b_0^0 + \beta b_1^0 + \gamma b_2^0 = 0 \; iif \; \alpha = 0 \text{ (for } t=0), \gamma = 0 \text{ (for } t=1), \beta = 0 \text{ (for } t=1/2)$$

Moreover, the dimension of $\mathcal{V}^1$ is 5, because V^1 is a basis, clearly by the symmetric relations and analogies between $V^0$and V^1.

We express now V^0 from V^1:

$$\begin{aligned}
b_1^0(t) = 2t(1-t) \quad &= (2t(1-2t) + 2t^2) \\
&= \tfrac{1}{2}(b_1^1(t) + b_2^1(t)) \text{ for } t \in [0;1/2[\\
b_1^0(t) = 2t(1-t) \quad &= (2(t-1)^2 + 2(2t-1)(1-t) \\
&= \tfrac{1}{2}(b_2^1(t) + b_3^1(t)) \text{ for } t \in [1/2;1[
\end{aligned}$$

So, because of the null values of $b_1^1(t)$ for $t \geq 1/2$ and $b_3^1(t)$ for $t \leq 1/2$,

$$b_1^0 = \frac{1}{2}(b_1^1 + b_2^1 + b_3^1)$$

We also have

$$\begin{aligned}
b_0^0(t) = (1-t)^2 &= 1 - 2t + t^2 \\
&= (1 - 4t + 4t^2) + (2t - 4t^2) + t^2 \\
&= (1-2t)^2 + 2t(1-2t) + t^2 \\
&= b_0^1(t) + \tfrac{1}{2}b_1^1(t) + \tfrac{1}{4}b_2^1(t) \text{ for } t \in [0; 1/2[\\
b_0^0(t) = (1-t)^2 &= (t-1)^2 \\
&= \tfrac{1}{4}b_2^1(t) \text{ for } t \in [1/2; 1[
\end{aligned}$$

So, because of the null values of $b_0^1(t)$ and $b_1^1(t)$ for $t \geq 1/2$,

$$b_0^0 = b_0^1 + \frac{1}{2}b_1^1 + \frac{1}{4}b_2^1$$

and, by symmetry, $b_2^0 = \frac{1}{4}b_2^1 + \frac{1}{2}b_4^1 + b_4^1$. Thus $V^0 \in \mathcal{V}^1$, $\mathcal{V}^0 \subset \mathcal{V}^1$.

We can also search for $\mathcal{W}^1$, such that $\mathcal{V}^1 = \mathcal{V}^0 \oplus \mathcal{W}^1$, $\mathcal{W}^1 = vect(W^1)$, $W^1 = \{b_3^0, b_4^0\}$.

We have linear spaces, with $\mathcal{V}^0 \subset \mathcal{V}^1$. So, from the linear algebra point of view, $\mathcal{V}^0 = A\mathcal{V}^1$, $V^0 = AV^1$,

$$\mathcal{W}^1 = \mathcal{V}^{0\perp} = A\mathcal{V}^{1\perp} = vect(Ker(A)V^1)$$

A is defined in the previous lines, by the expression of V^0 from V^1:

$$A = \begin{pmatrix} 1 & 1/2 & 1/4 & 0 & 0 \\ 0 & 1/2 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/4 & 1/2 & 1 \end{pmatrix} \quad Ker(A) = \begin{pmatrix} 0 & -1 \\ 1 & 4 \\ -2 & -4 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So, the vectors of $Ker(A)$ are a basis of $\mathcal{W}^1$, expressed in the V^1 basis of $\mathcal{V}^1$. These vectors are

$$\begin{aligned}
&b_1^1(t) - 2b_2^1(t) + b_3^1(t) \\
&= 4t(1-2t) - 8t^2 && = 4t(1-4t) \text{ for } t \in [0; 1/2[\\
&= -8(t-1)^2 + 4(2t-1)(1-t) && = 4(1-t)(4t-3) \text{ for } t \in [1/2; 1[
\end{aligned}$$

$$\begin{aligned}
&-b_0^1(t) + 4b_1^1(t) - 4b_2^1(t) + b_4^1(t) \\
&= -(1-2t)^2 + 16t(1-2t) - 16t^2 && = -1 + 20t - 52t^2 \text{ for } t \in [0; 1/2[\\
&= -16(t-1)^2 + (2t-1)^2 && = (3-2t)(6t-5) \text{ for } t \in [1/2; 1[
\end{aligned}$$

In this vector space, we can select the following definitions for b_3^0 and b_4^0:

$$
\begin{aligned}
b_3^0(t) &= 2t(4t-1) \text{ for } t \in [0; 1/2[\\
&= 2(t-1)(4t-3) \text{ for } t \in [1/2; 1[
\end{aligned}
$$

$$
\begin{aligned}
b_4^0(t) &= (t-1/2)(-10t+1) \text{ for } t \in [0; 1/2[\\
&= (t-1/2)(10t-9) \text{ for } t \in [1/2; 1[
\end{aligned}
$$

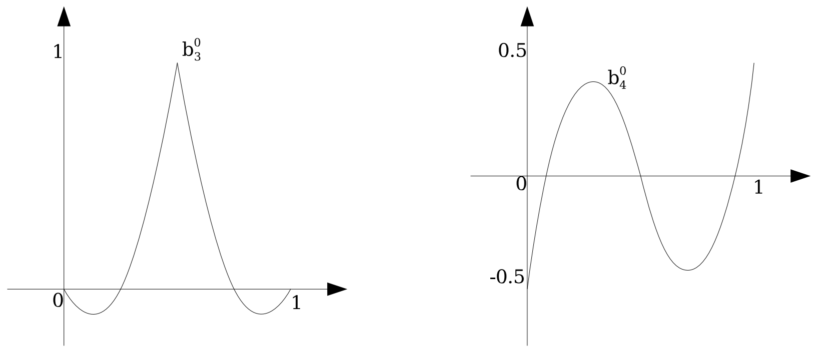

Figure 3: Basis of $\mathcal{W}^1$.