Truncation Error in Image Interpolation

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GREYC

cnrs dépasser les frontières

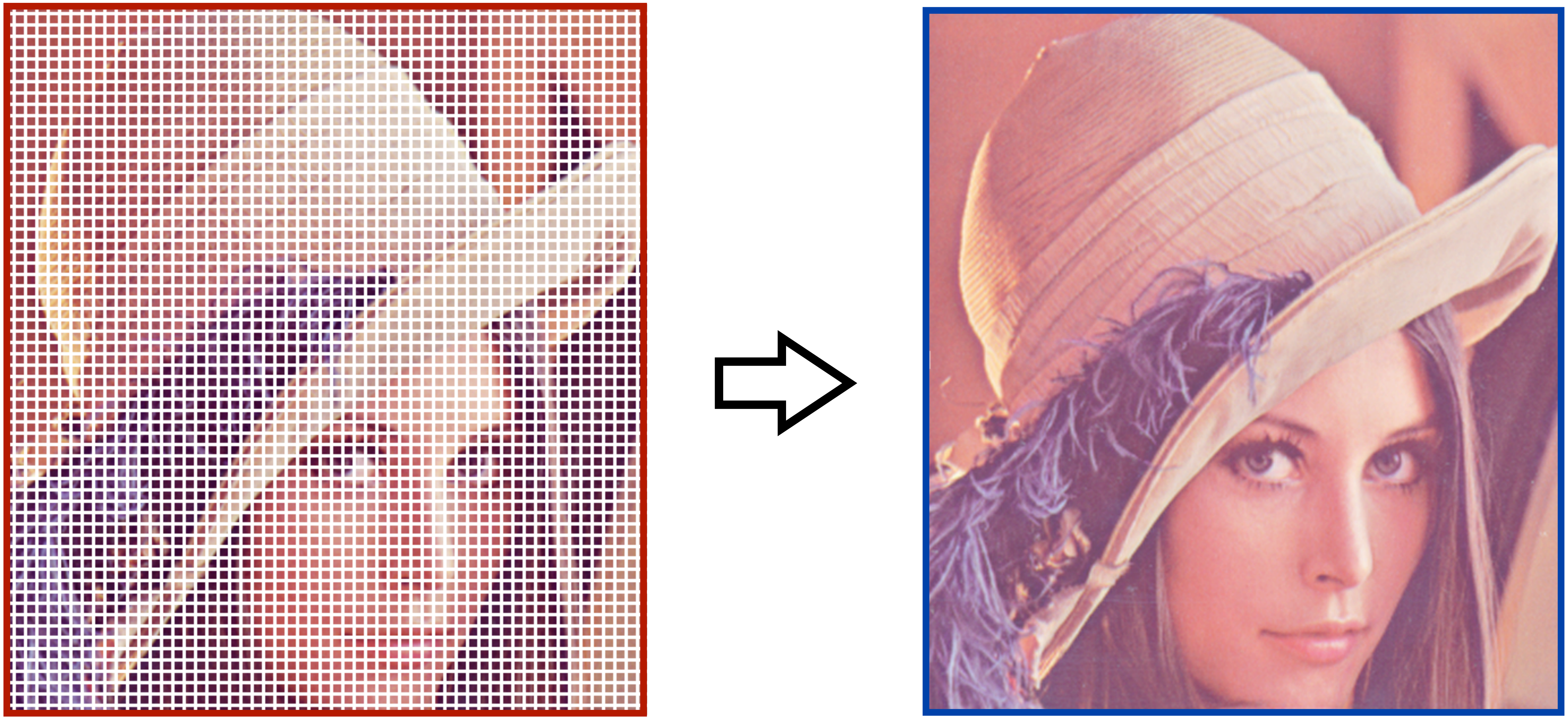
ENSICAEN
ÉCOLE NATIONALE SUPÉRIEURE D'INGÉNIEURS DE CAEN
& CENTRE DE RECHERCHE

Collaborator

Jean-Michel Morel

Truncation error:

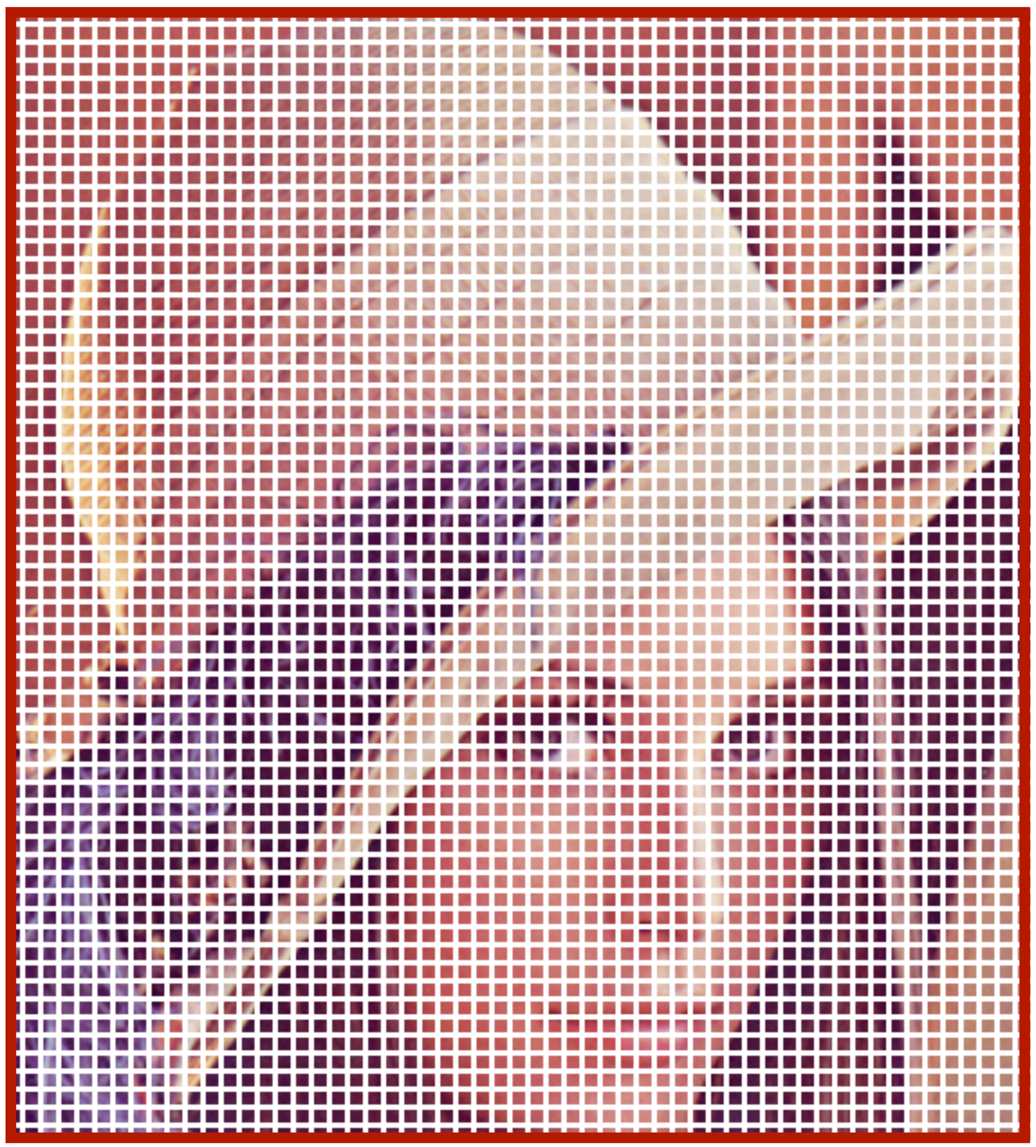
What is that?

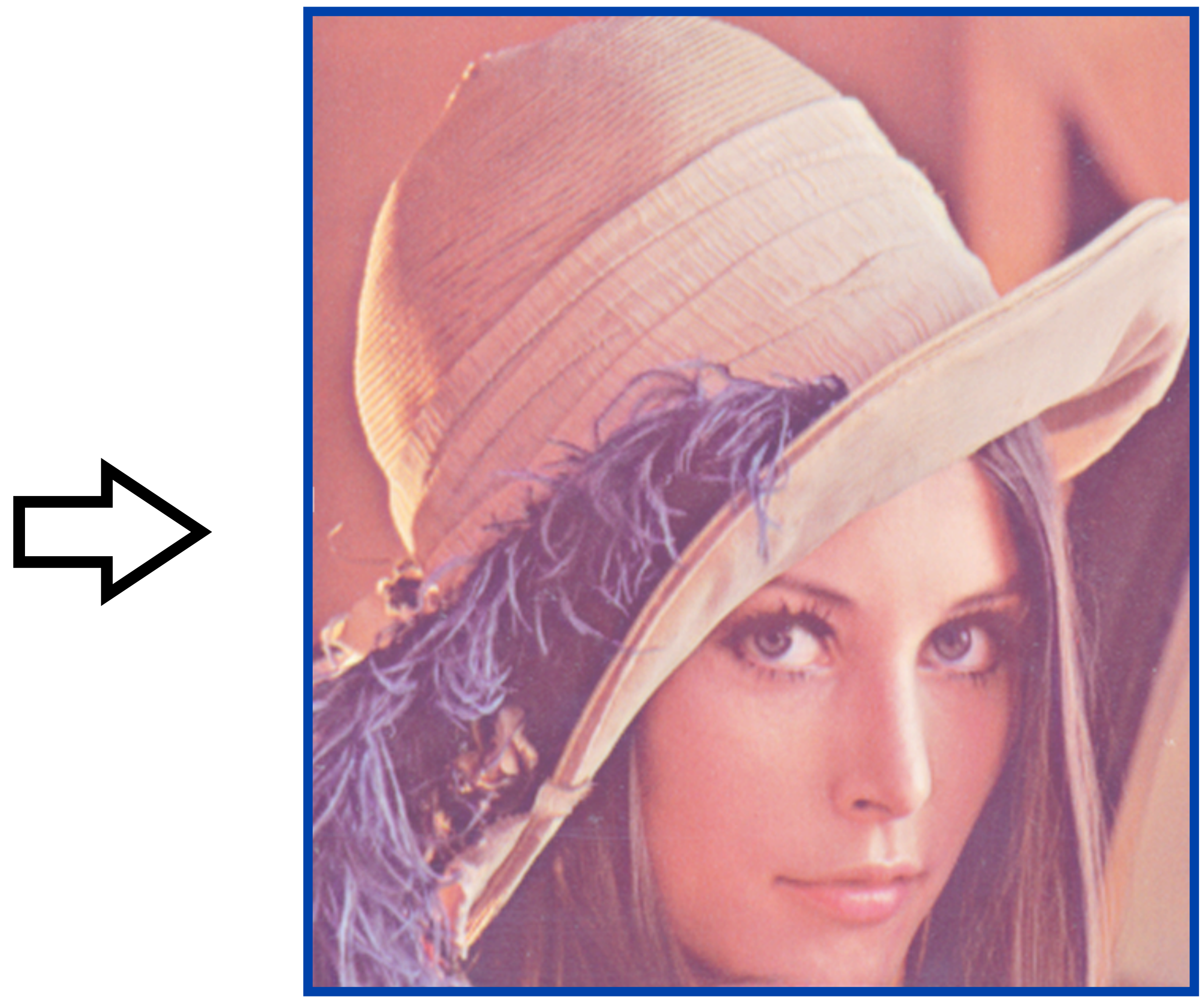


X_k's $\quad$ X_t's

Truncation error:

What is that?

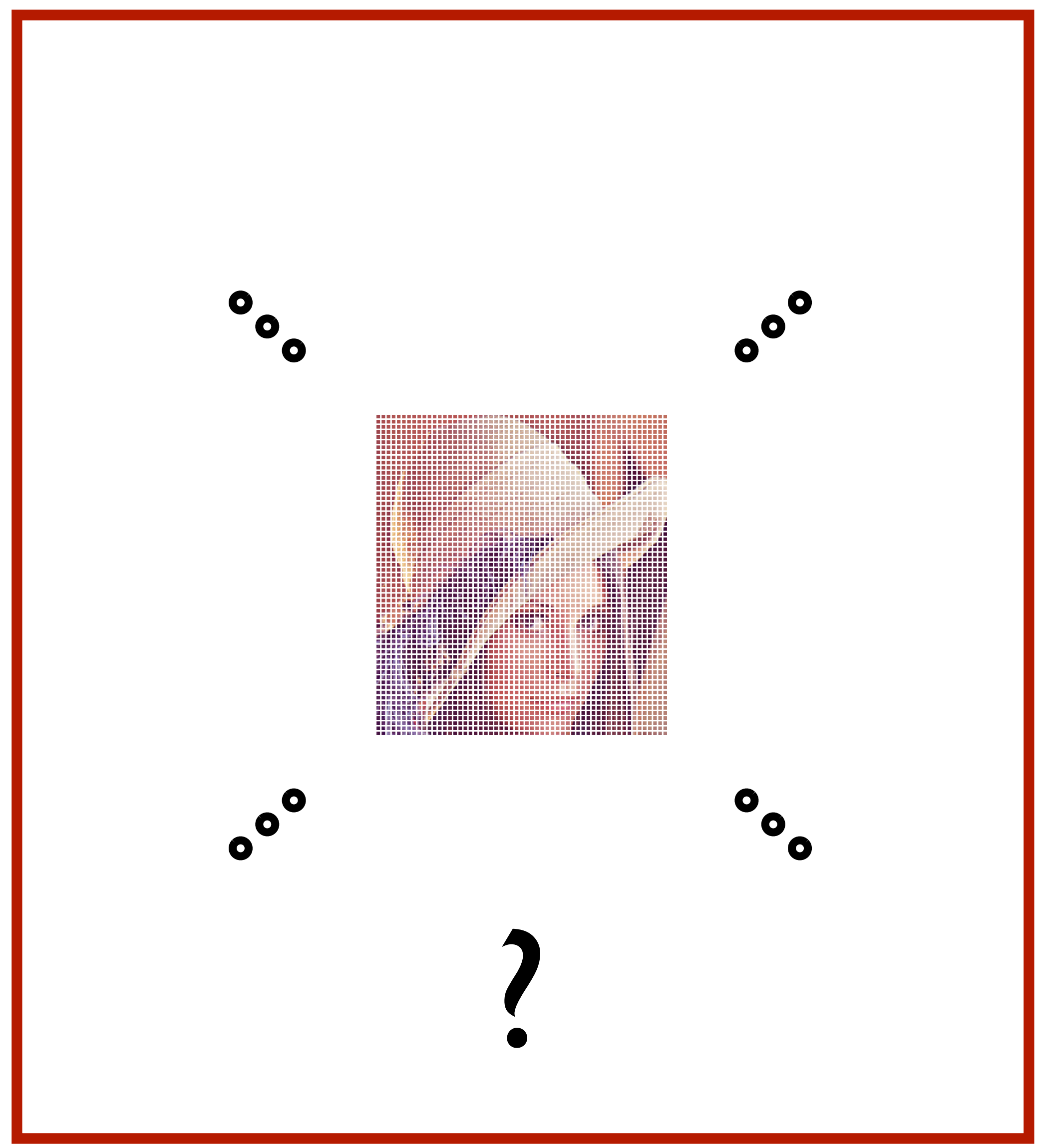




$$X_t := \sum_{k \in \mathbb{Z}^2} X_k \operatorname{sinc}(t - k)$$

Truncation error:

What is that?

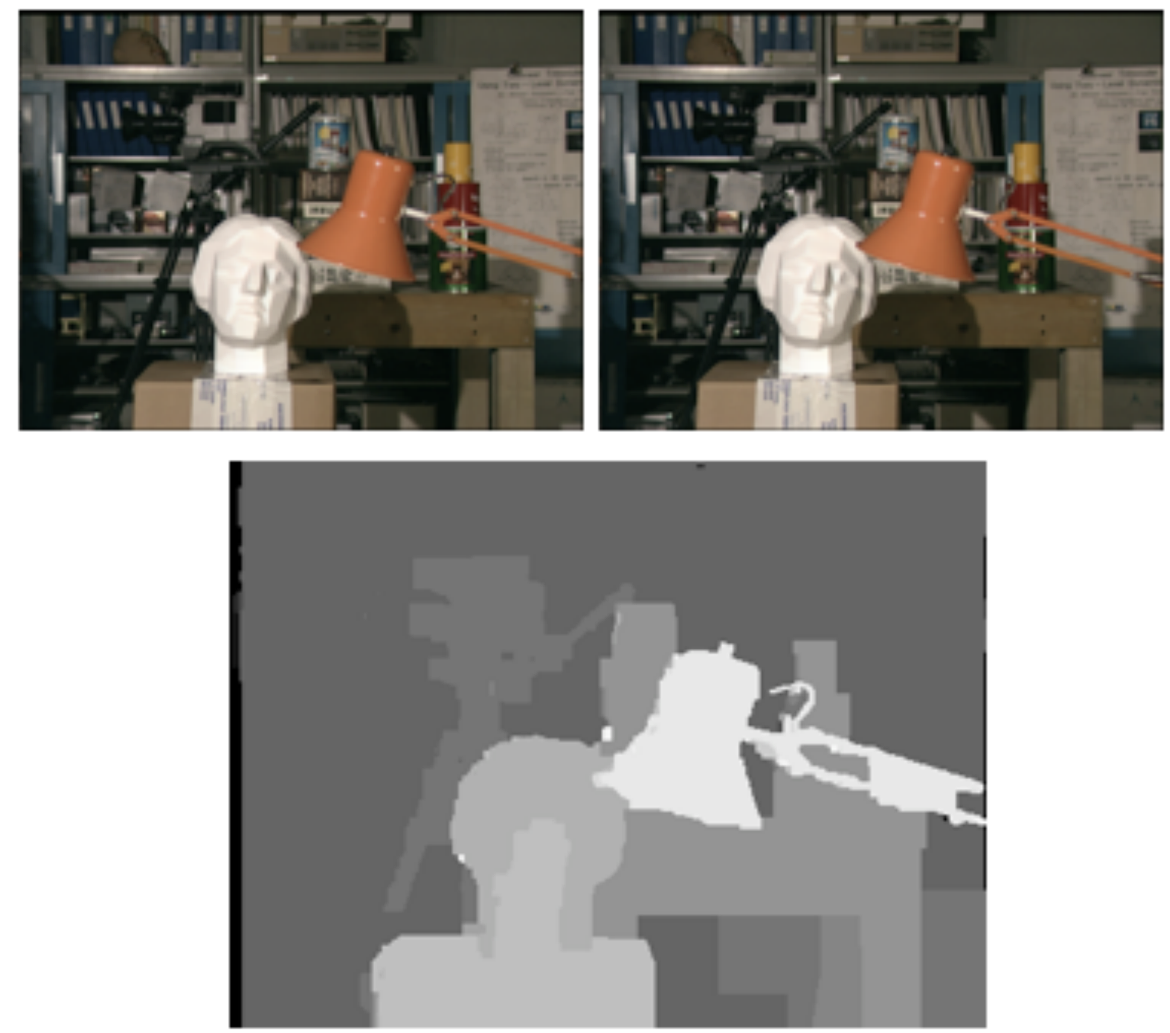


Context

- Motivations
- Assumptions
- Goal
- Related work

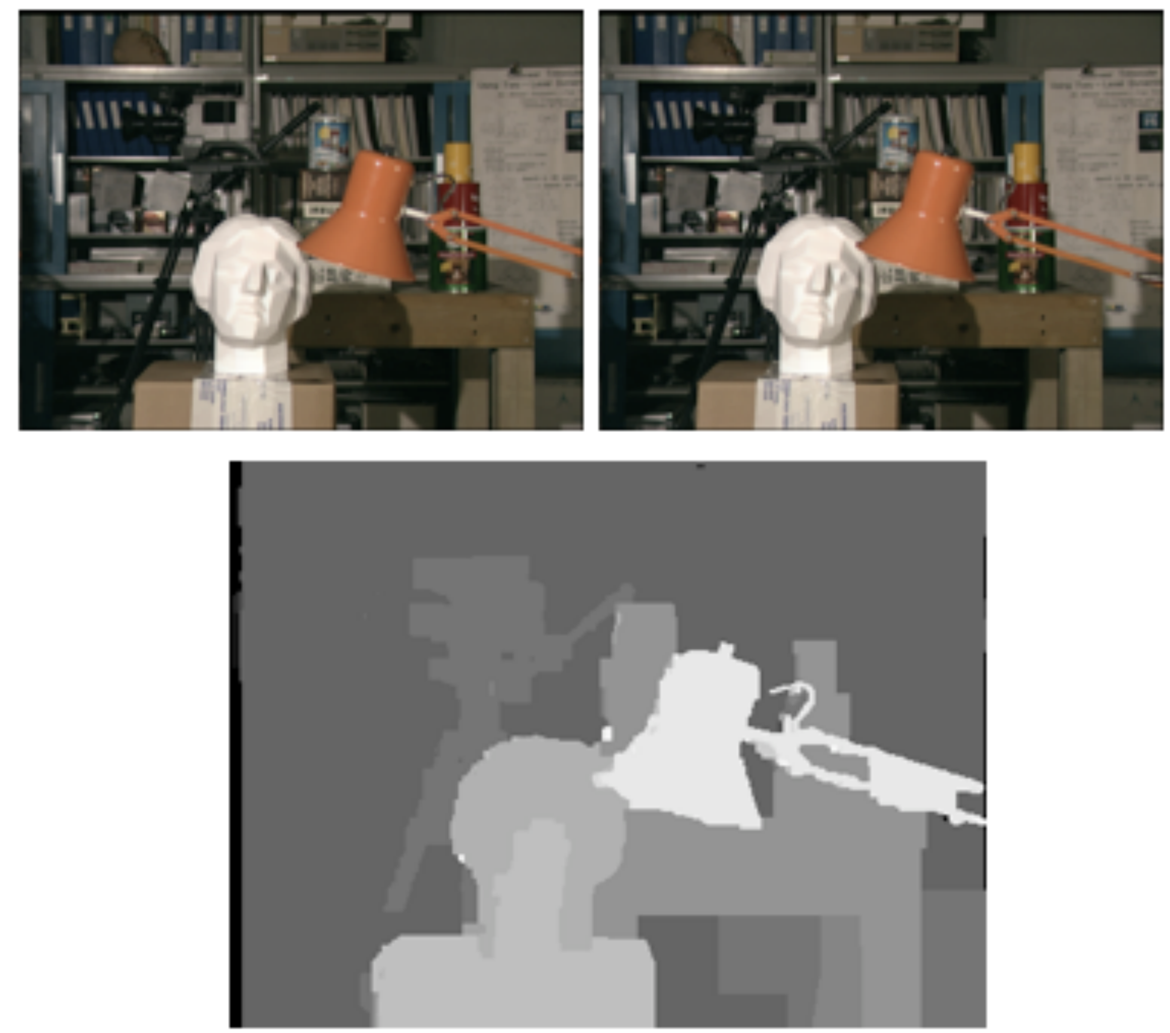
Motivations

- Image registration
  - optical flow
  - stereopsis
  - super-resolution
- sub-pixel accuracy

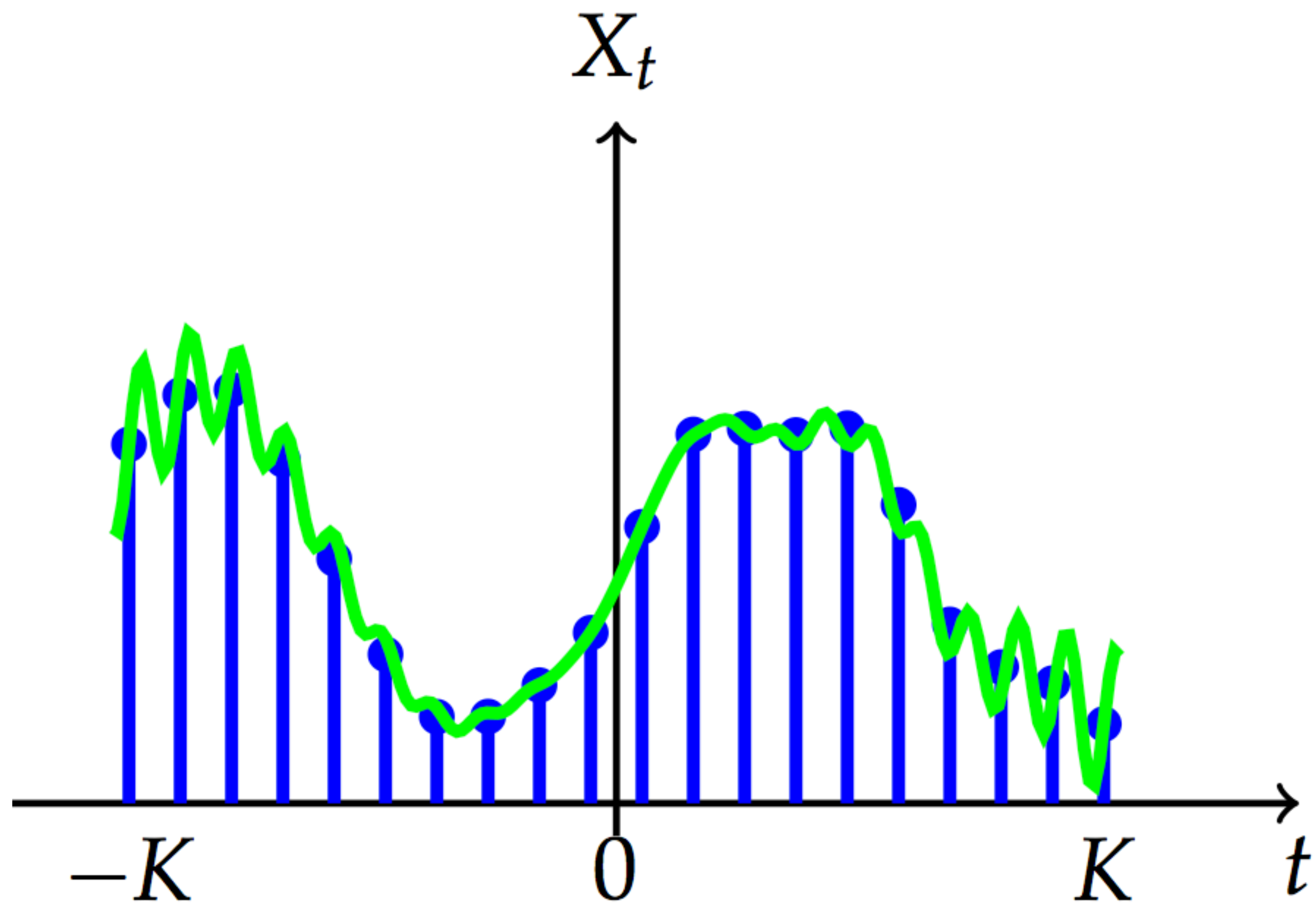


Motivations

- Image registration
  - optical flow
  - stereopsis
  - super-resolution
- sub-pixel accuracy
- error ~ **quantization**



Assumptions



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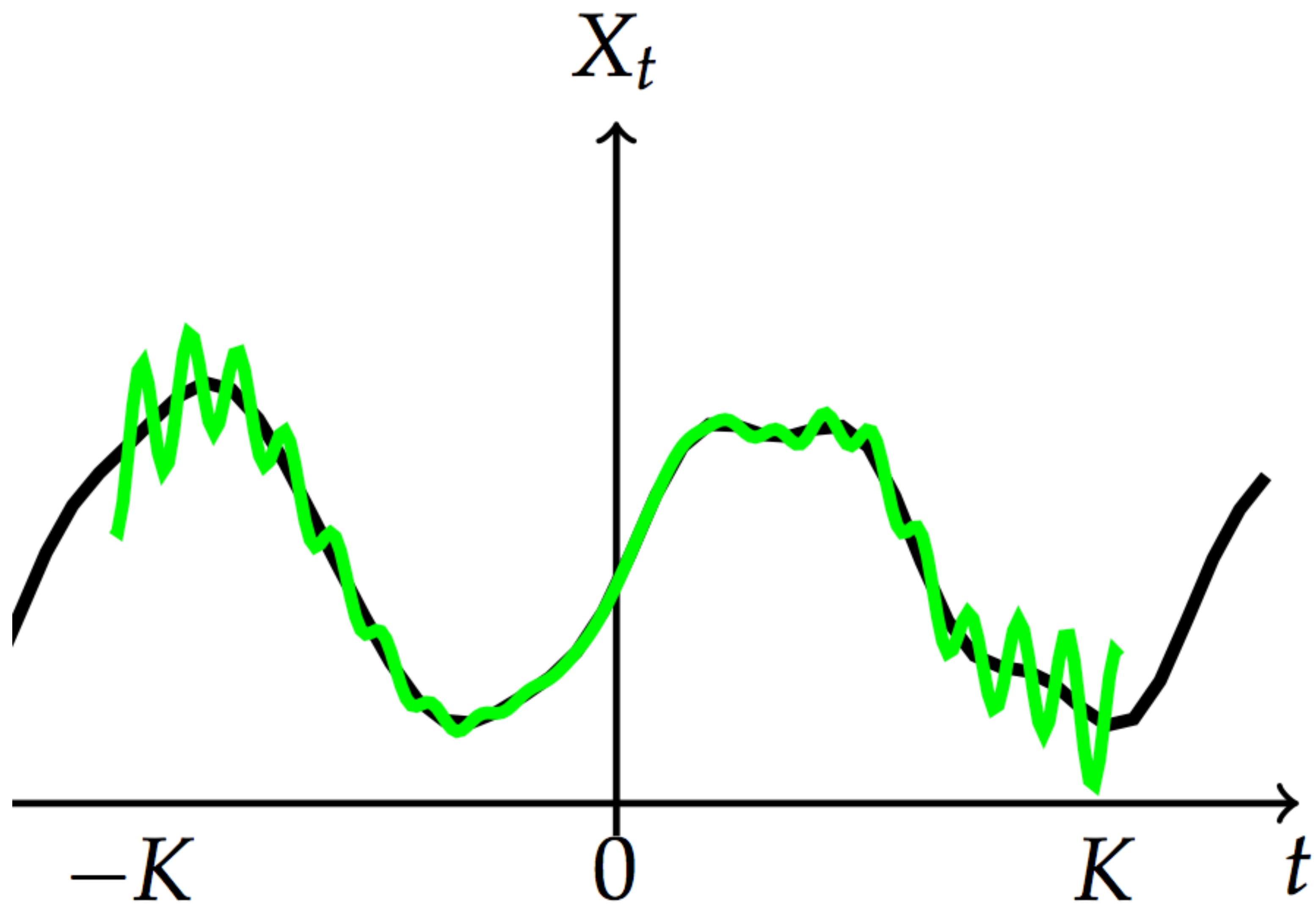
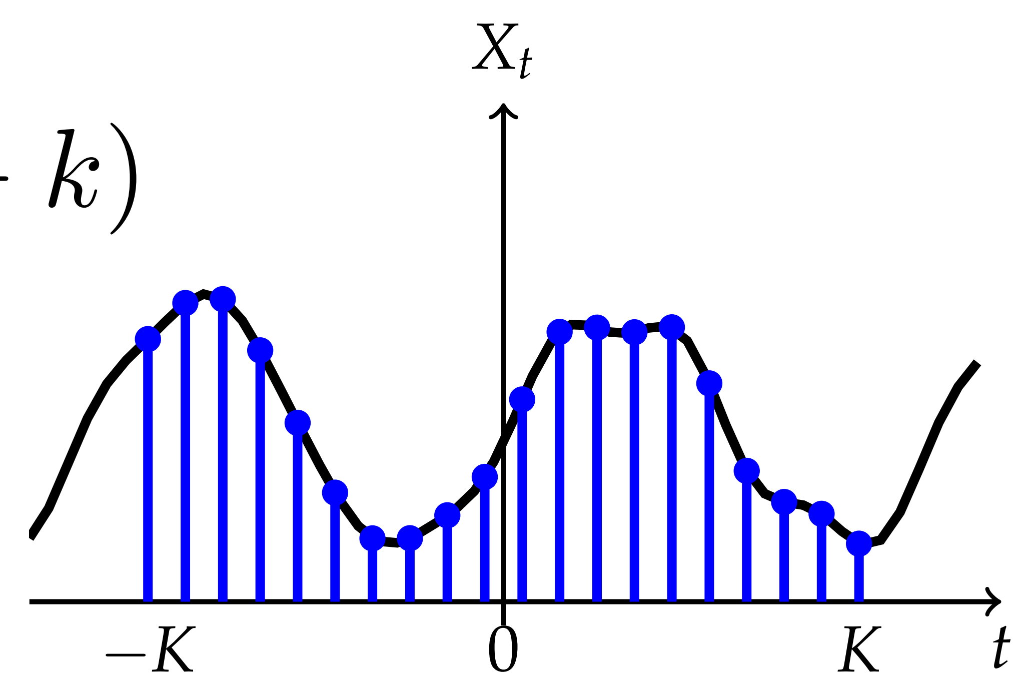


Assumptions

- $X_t \ (t \in \mathbb{R})$ a 1d random process
  - observed on $k \in \{-K, \ldots, K\}$
  - weakly **stationary** $\mu, d\Psi_X(\omega)$
  - **no aliasing**

$$X_t := \sum_{k \in \mathbb{Z}} X_k \mathrm{sinc}(t-k)$$



Goal

- Linear shift-invariant

$$\tilde{X}_t := \sum_{k \leq K} X_k h(t-k)$$

- Practical bounds on

$$RMSE[\tilde{X}_t] := \sqrt{\mathbb{E}\left[(\tilde{X}_t - X_t)^2\right]}$$

Goal

- Linear shift-invariant

$$\tilde{X}_t := \sum_{k \leq K} X_k h(t-k) \qquad \begin{cases} h(t) = \mathrm{sinc}(t) \\ h(t) = \mathrm{sincd}_K(t) \end{cases}$$

- Practical bounds on

$$RMSE[\tilde{X}_t] := \sqrt{\mathbb{E}\left[(\tilde{X}_t - X_t)^2\right]}$$

➡ Sinc interpolation ➡ DFT interpolation

Related Work

Truncation Error		Approximation	
Jagerman	1966	Strang & Fix	1971
Yao & Thomas	1966	Blu & Unser	1999
Campbell	1968	Condat & al.	2005
Brown	1969		
Xu & Huang & Li	2009		

Other

Jerri	1977	Moisan	2011

Related Work

Truncation Error

Jagerman	1966
Yao & Thomas	1966
Campbell	1968
Brown	1969
Xu & Huang & Li	2009

➡ sinc only ➡ oversampled case

Approximation

Strang & Fix	1971
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Truncation Error

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Approximation

Strang & Fix	1971
Blu & Unser	1999
Condat & al.	2005

➡ $K = \infty$

Other

Jerri	1977
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Related Work

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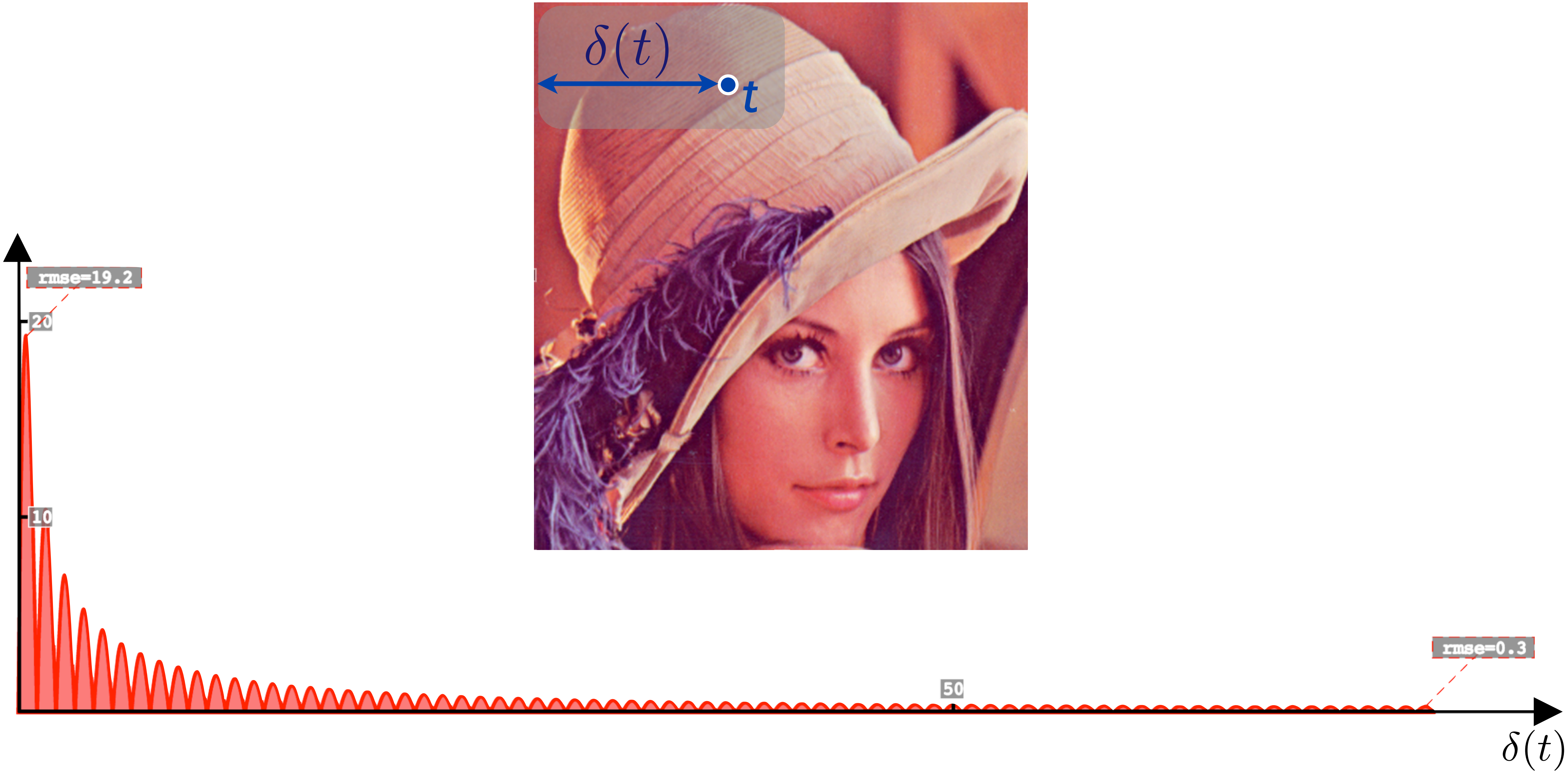
Rest of the talk

- Theoretical bounds
- Experimental results
- Discussion & conclusion

A bit of intuition...



Theoretical bounds

$$MSE[\tilde{X}](t) = \frac{\sin^2(\pi t)}{\pi^2} \times \left(\begin{array}{c} \mu^2 \mathcal{O}\left(\frac{1}{\delta(t)^2}\right) \\ + \\ {\sigma'}_{\alpha}^{2} \mathcal{O}\left(\frac{1}{\delta(t)^2}\right) \\ + \\ \sigma_{\alpha}^{2} \mathcal{O}\left(\frac{1}{\delta(t)}\right) \end{array} \right)$$

Theoretical bounds

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Theoretical bounds

$$MSE[\tilde{X}](t) = \frac{\sin^2(\pi t)}{\pi^2} \times \begin{pmatrix} 0\mu^2\mathcal{O}\left(\frac{1}{\delta(t)^2}\right) \\ + \\ 2{\sigma'}_{\alpha}^{2}\mathcal{O}\left(\frac{1}{\delta(t)^2}\right) \\ + \\ 2\sigma_{\alpha}^{2}\mathcal{O}\left(\frac{1}{\delta(t)}\right) \end{pmatrix}$$

➡ DFT modifications

Spectral representation

$$MSE[\tilde{X}](t) = \underbrace{\mu^2 \left| 1 - \sum_{|k| \le K} h(t-k) \right|^2}_{MSE[\mu](t)}$$

$$+$$

$$\underbrace{\frac{1}{2\pi} \int \left| e^{i\omega t} - \sum_{|k| \le K} e^{i\omega k} h(t-k) \right|^2 d\Psi_X(\omega)}_{MSE[d\Psi_X](t)}$$

➡ Average component ➡ Spectral component

Spectral representation

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➡ Aliasing is not forbidden

Spectral representation

$$MSE[\tilde{X}](t) = \underbrace{\mu^2 \left| \sum_{k \in \mathbb{Z}} \mathrm{sinc}(t-k) - \sum_{|k| \le K} h(t-k) \right|^2}_{MSE[\mu](t)}$$

$$+$$

$$\underbrace{\frac{1}{2\pi} \int_{|\omega| \le \pi} \left| \sum_{k \in \mathbb{Z}} e^{i\omega k} \mathrm{sinc}(t-k) - \sum_{|k| \le K} e^{i\omega k} h(t-k) \right|^2 d\Psi_X(\omega)}_{MSE[d\Psi_X](t)}$$

➡ Under no aliasing condition

Spectral representation

$$MSE[\tilde{X}](t) = \underbrace{\mu^2 \left| \sum_{k\in\mathbb{Z}} \mathrm{sinc}(t-k) - \sum_{|k|\leq K} \mathrm{sinc}(t-k) \right|^2}_{MSE[\mu](t)}$$

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➡ Under no aliasing condition ➡ Sinc

Average component

$$MSE[\mu](t) = \frac{\sin^2(\pi t)}{\pi^2}\mu^2\mathcal{O}\left(\frac{1}{\delta(t)^2}\right)$$

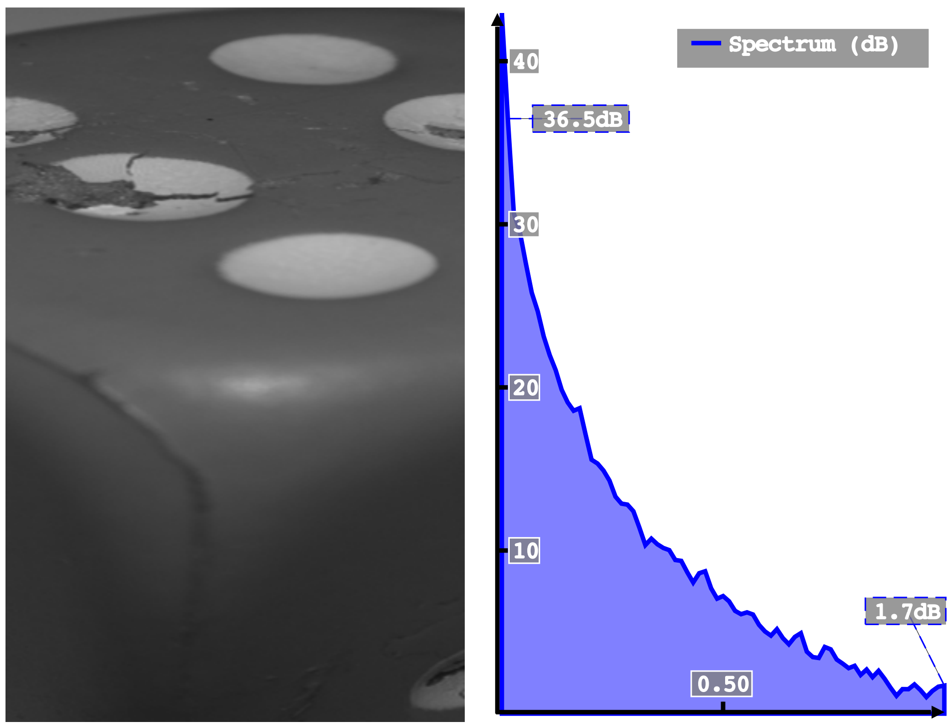
➡ Gibbs phenomenon

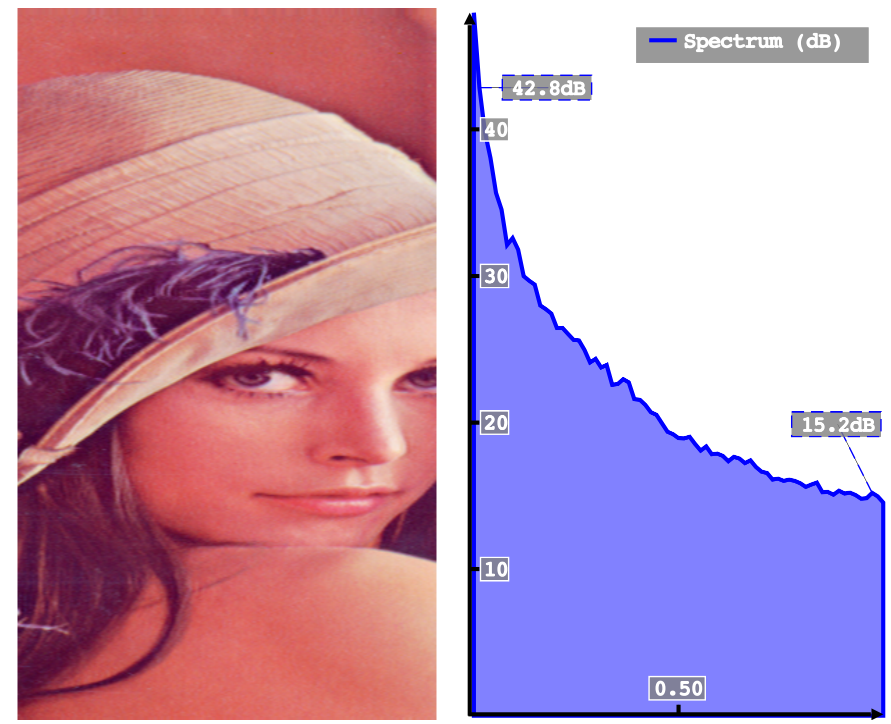
Average component

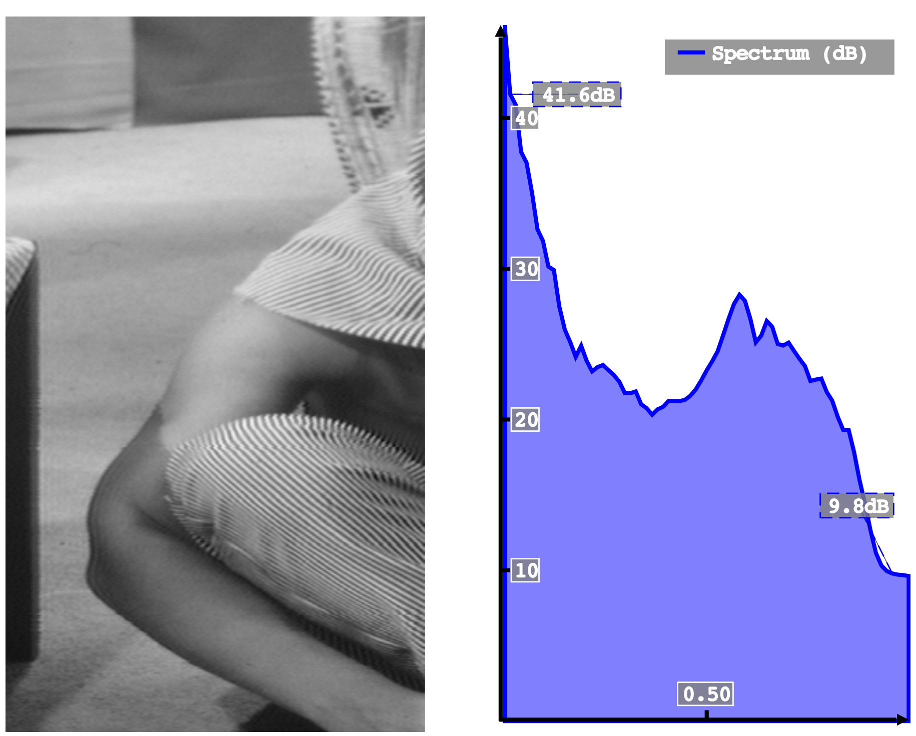
$$MSE[\mu](t) = 0$$

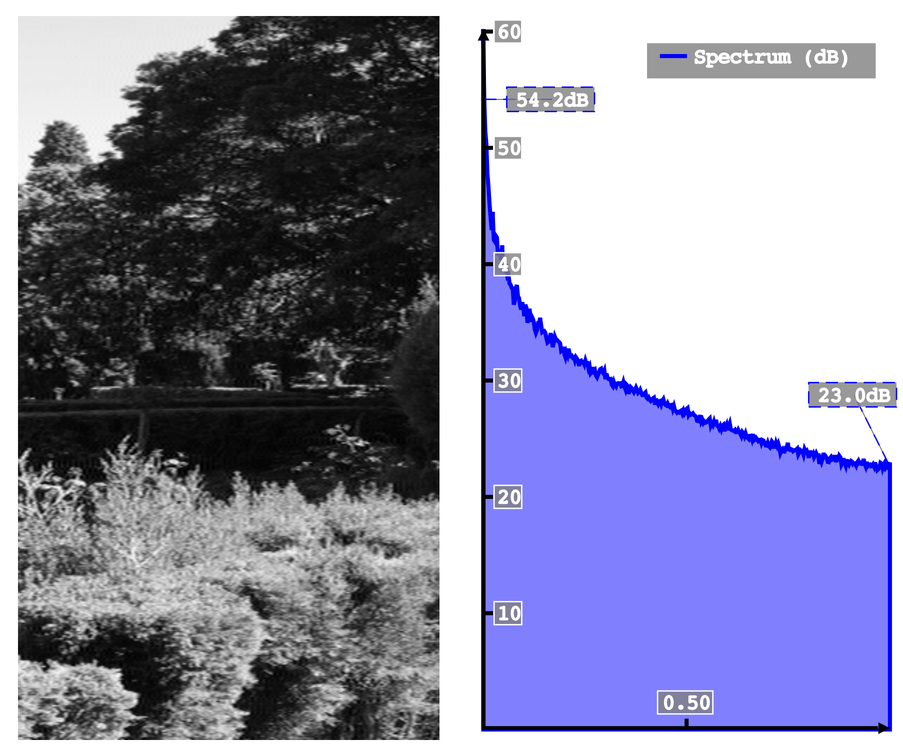
➡ DFT

Spectral component

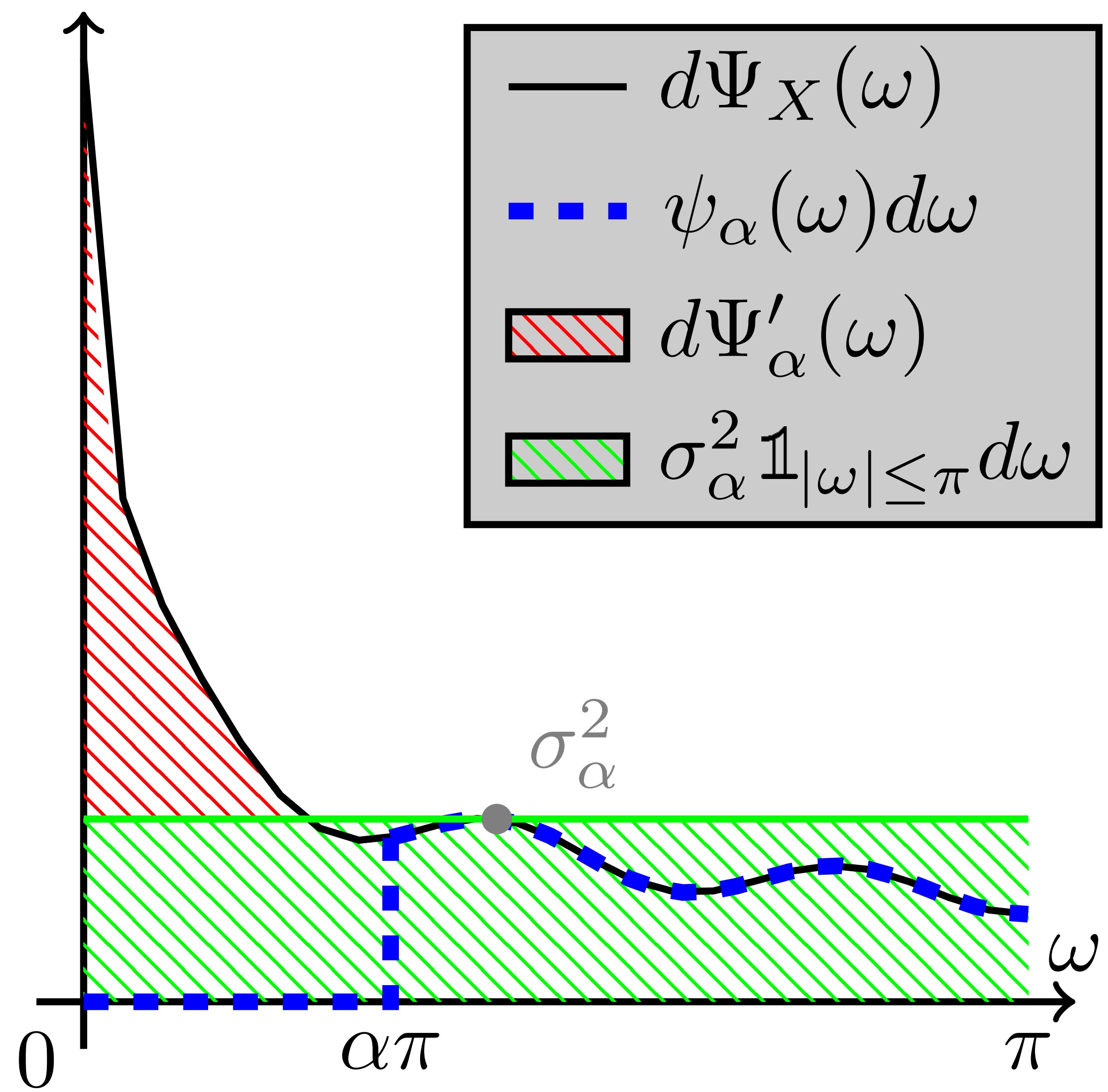








Spectral decomposition



➡ spectrum ≤ oversampled + white-noise

Oversampled case

$$\text{supp}(d\Psi'_\alpha) \subset \{|\omega| \leq \alpha\pi\}$$

$$\Longrightarrow$$

$$MSE[d\Psi'_\alpha](t) = \frac{\sin^2(\pi t)}{\pi^2}\sigma'^2_\alpha \mathcal{O}\left(\frac{1}{\delta(t)^2}\right)$$

where,

$$\sigma'^2_\alpha = \frac{1}{\pi}\int_{|\omega|\leq\alpha\pi} \frac{1}{1+\cos(\omega)} d\Psi'_\alpha(\omega)$$

Oversampled case

$$\mathrm{supp}(d\Psi'_\alpha) \subset \{|\omega| \leq \alpha\pi\}$$

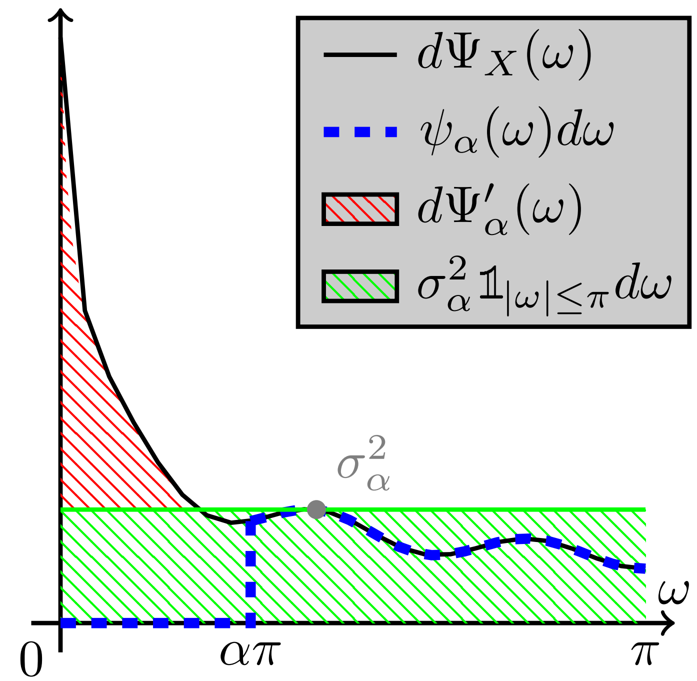
$$\Longrightarrow$$

$$\left\{ \begin{array}{l} MSE[d\Psi'_\alpha](t) \\ MSE[\mu](t) \end{array} \right\} = \frac{\sin^2(\pi t)}{\pi^2} \left\{ \begin{array}{l} \sigma'^2_\alpha \\ \mu^2 \end{array} \right\} \mathcal{O}\left(\frac{1}{\delta(t)^2}\right)$$

where,

$$\sigma'^2_\alpha = \frac{1}{\pi} \int_{|\omega| \leq \alpha\pi} \frac{1}{1+\cos(\omega)} d\Psi'_\alpha(\omega)$$

White-noise



$$d\Psi(\omega) = \sigma_\alpha^2 \mathbb{1}_{|\omega|\leq\pi} d\omega$$

$$\Longrightarrow$$

$$MSE[d\Psi](t) = \frac{\sin^2(\pi t)}{\pi^2} \sigma_\alpha^2 \mathcal{O}\left(\frac{1}{\delta(t)}\right)$$

➡ Slow decay

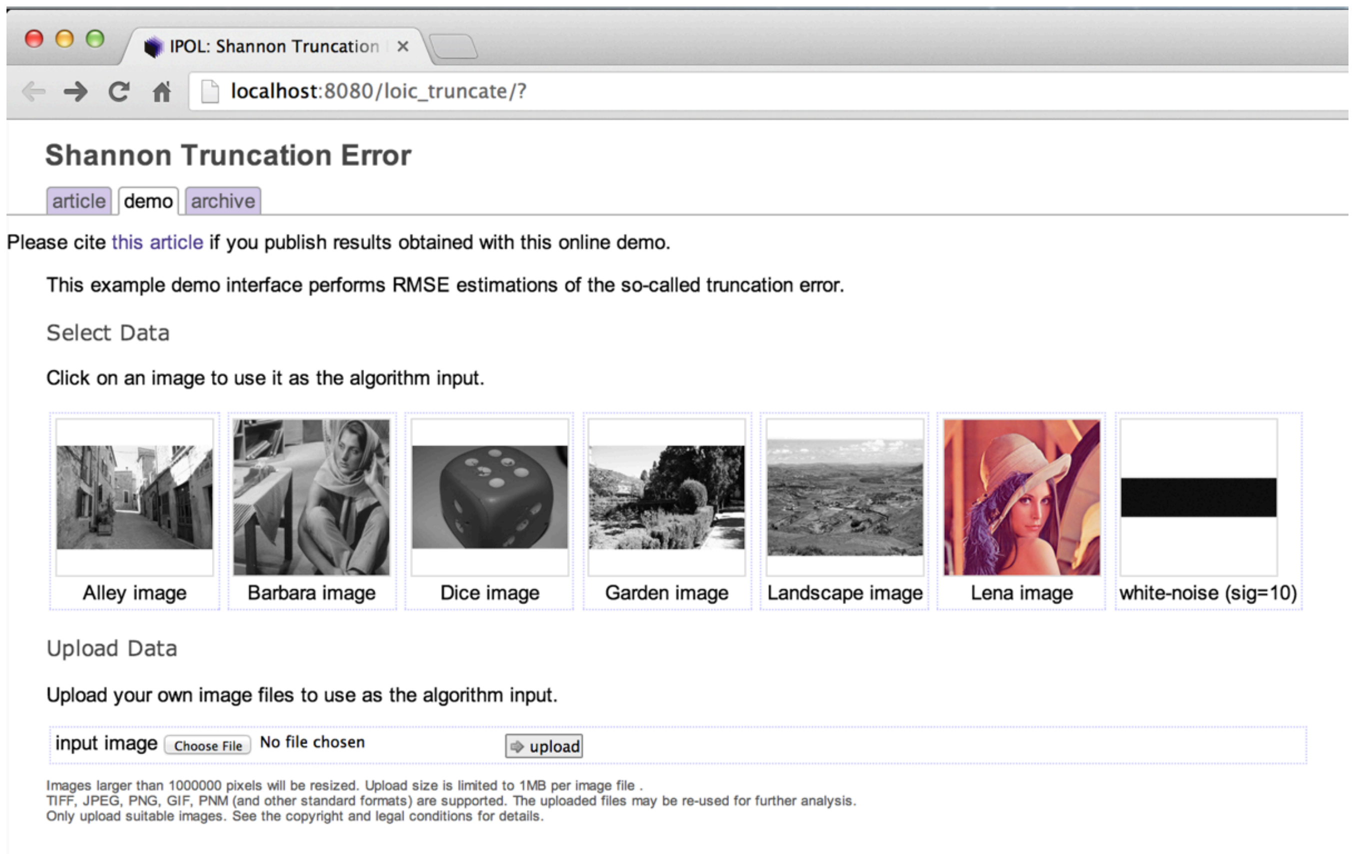
Recap

$$MSE[\tilde{X}](t) = \frac{\sin^2(\pi t)}{\pi^2} \times \left(\begin{array}{c} 0\mu^2 \mathcal{O}\left(\frac{1}{\delta(t)^2}\right) \\ + \\ 2{\sigma'}_\alpha^2 \mathcal{O}\left(\frac{1}{\delta(t)^2}\right) \\ + \\ 2\sigma_\alpha^2 \mathcal{O}\left(\frac{1}{\delta(t)}\right) \end{array} \right)$$

➡ DFT modifications

Experimental results

- Bound validity
- Bound tightness
- Order of magnitude



➡ online demo: IPOL · Image Processing On Line

Experimental results

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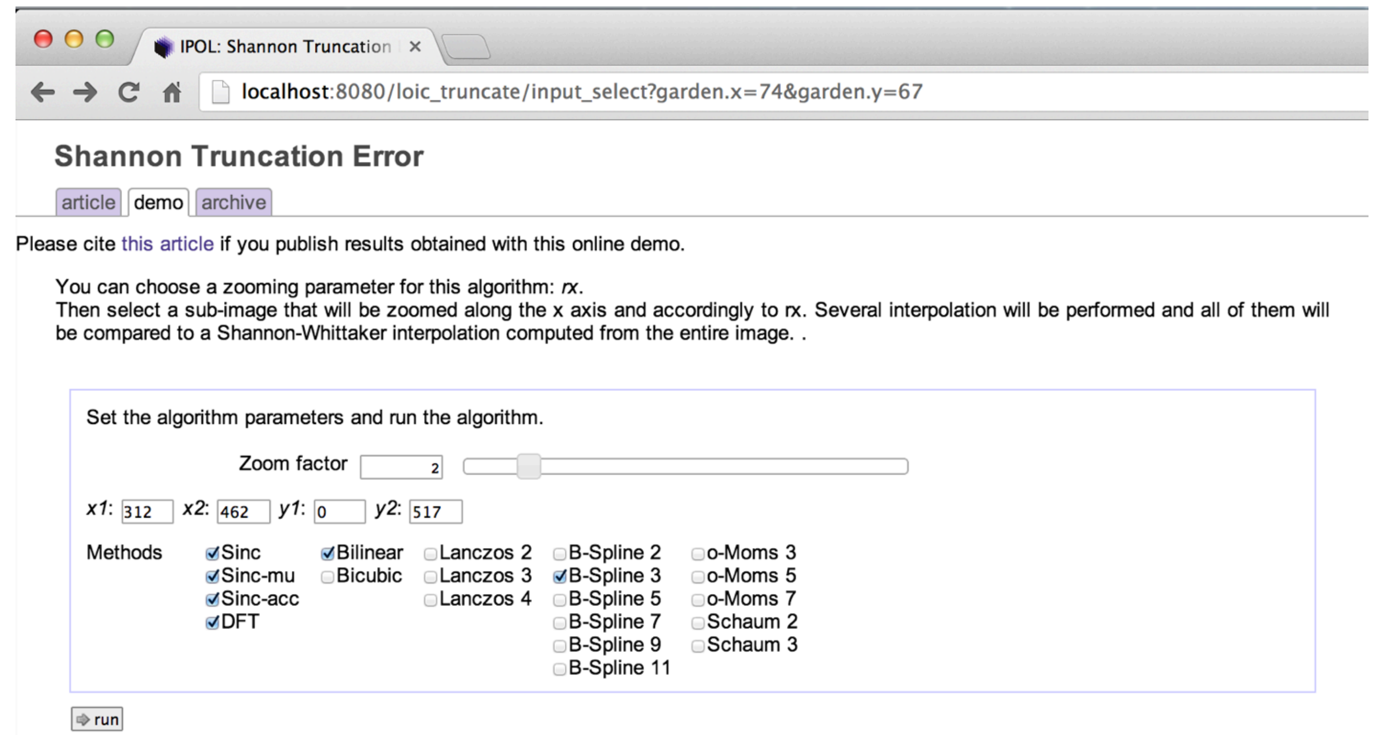

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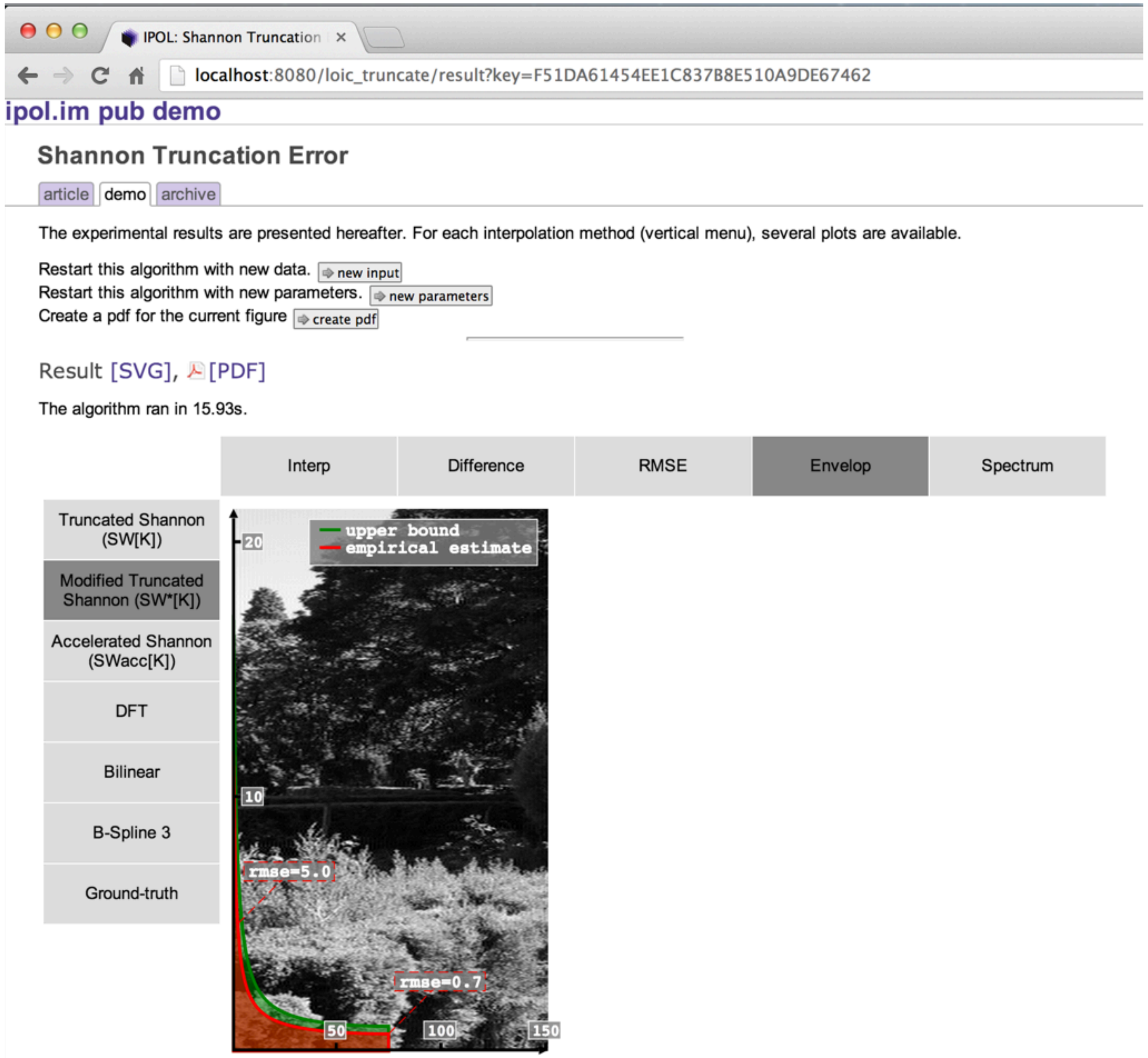

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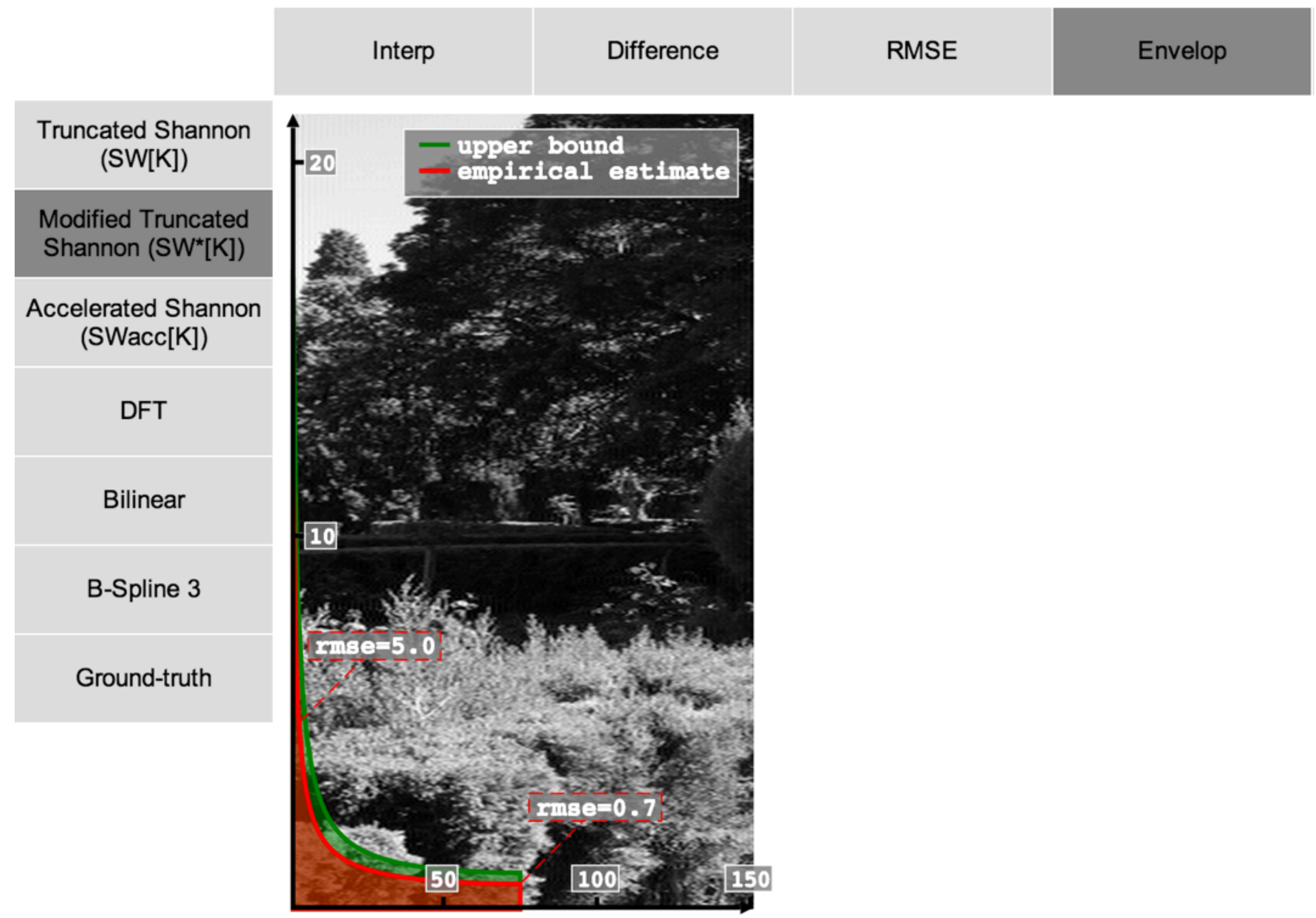

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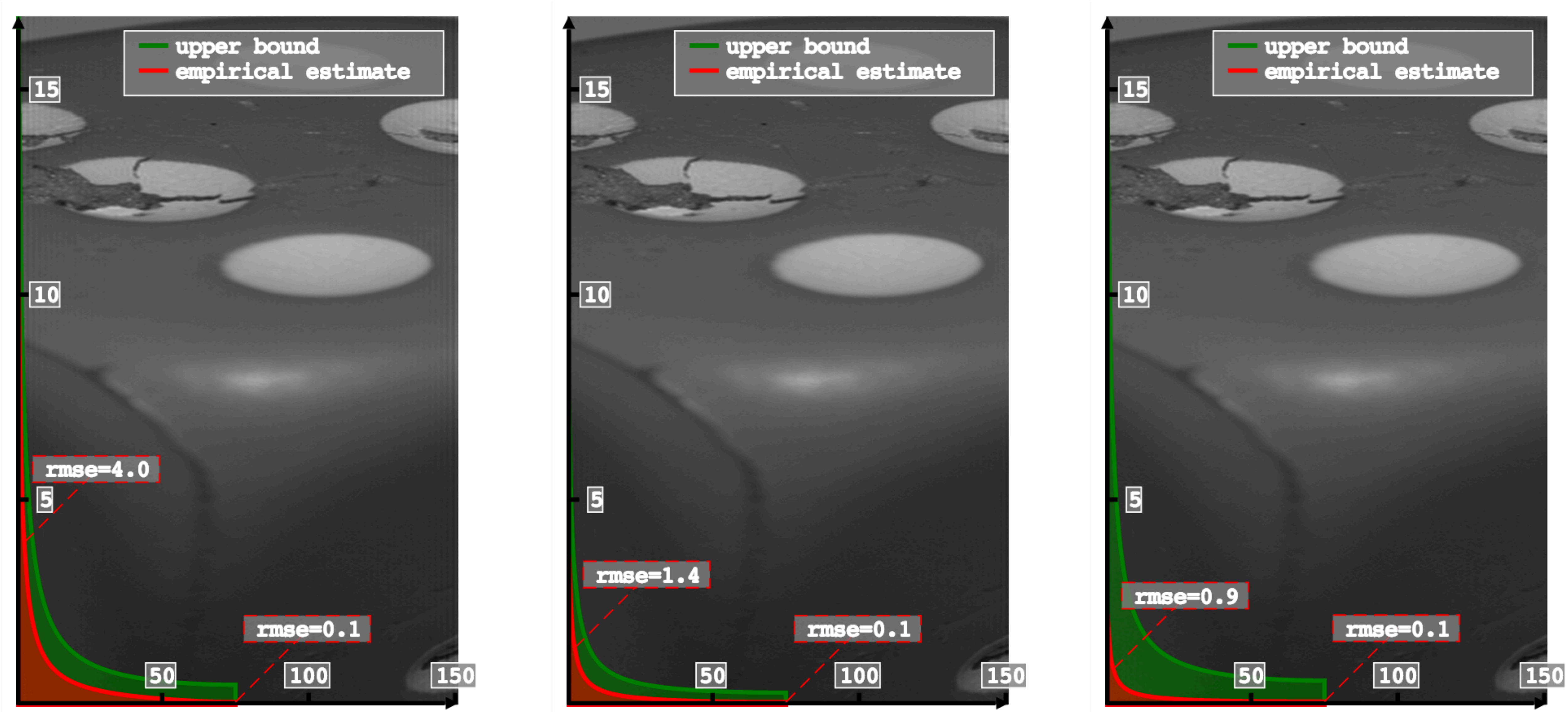
Bound tightness

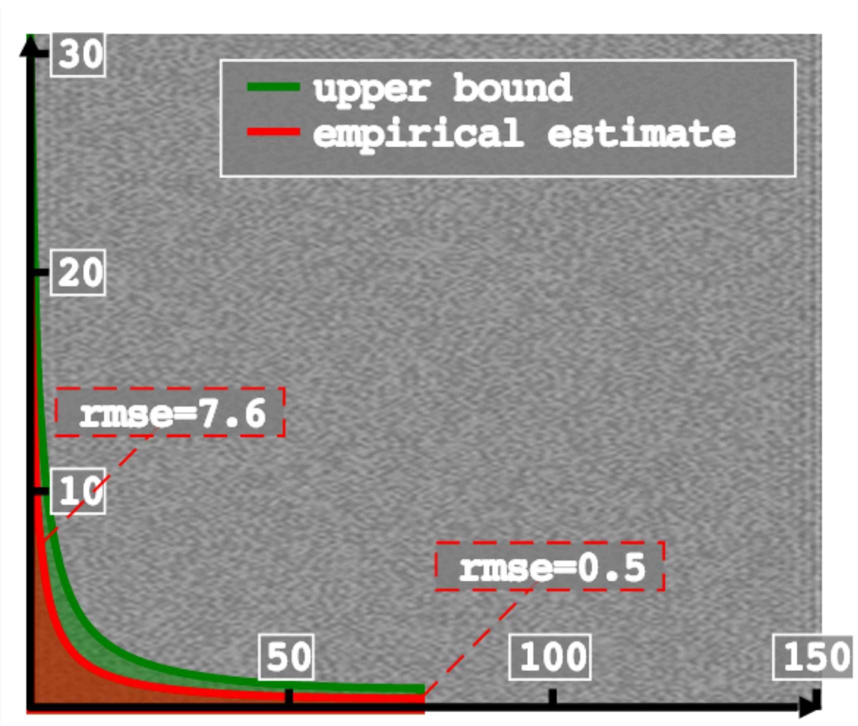


Sinc — Sinc w/o μ — DFT

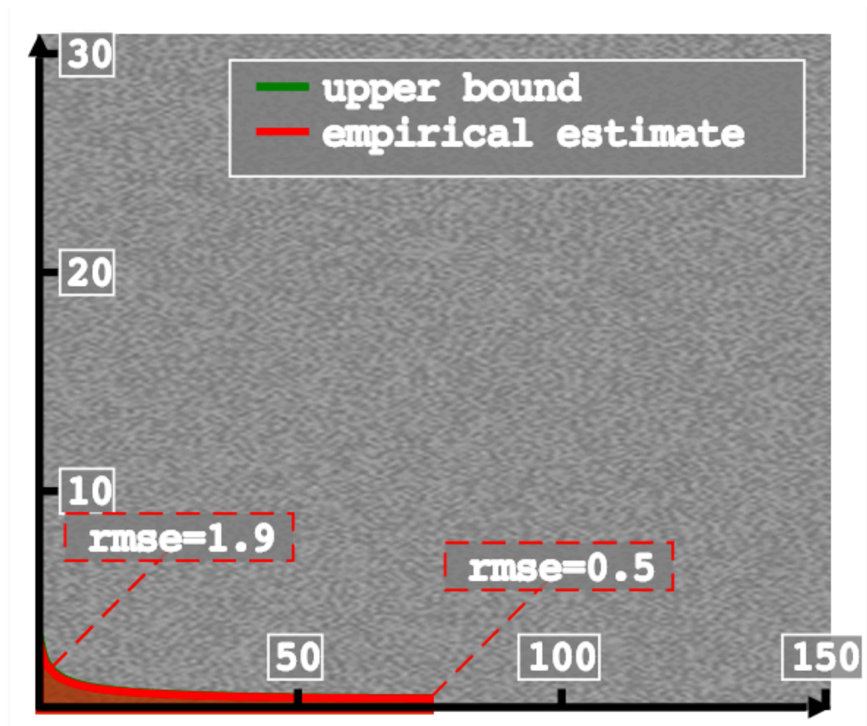
➡ Smooth image

➡ $\sqrt{\mathbb{E}[\text{quant}^2]} = 0.3$

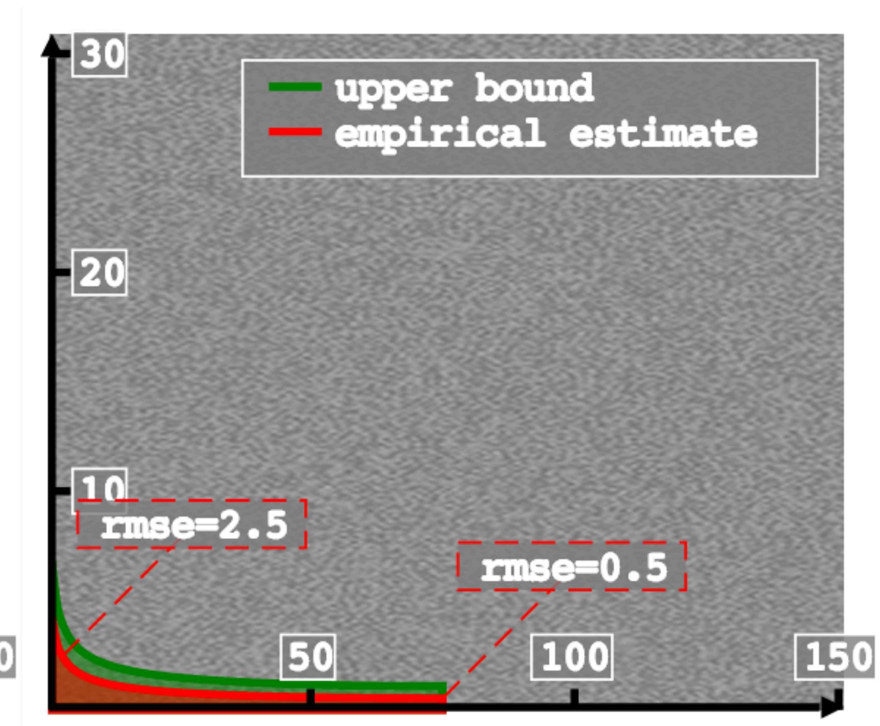
Bound tightness



Sinc



Sinc w/o μ



DFT

➡ Simulated white-noise

➡ $\sqrt{\mathbb{E}[\text{quant}^2]} = 0.3$

Bound tightness

Sinc

Sinc w/o μ

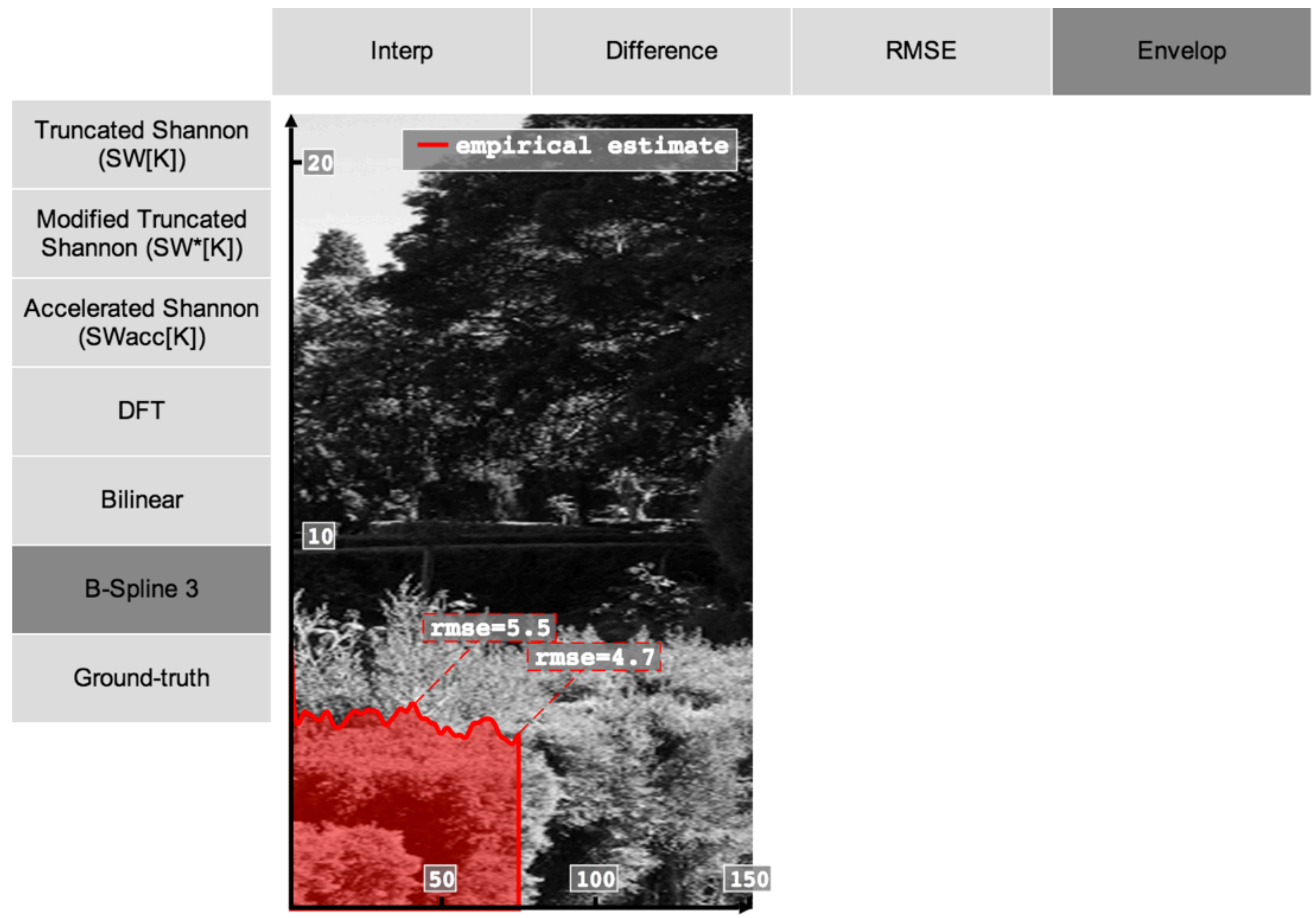
DFT

➡ Textured image

➡ $\sqrt{\mathbb{E}[\text{quant}^2]} = 0.3$

Other kernels?

- ~~Bound validity~~
- ~~Bound tightness~~
- <u>Order of magnitude</u>



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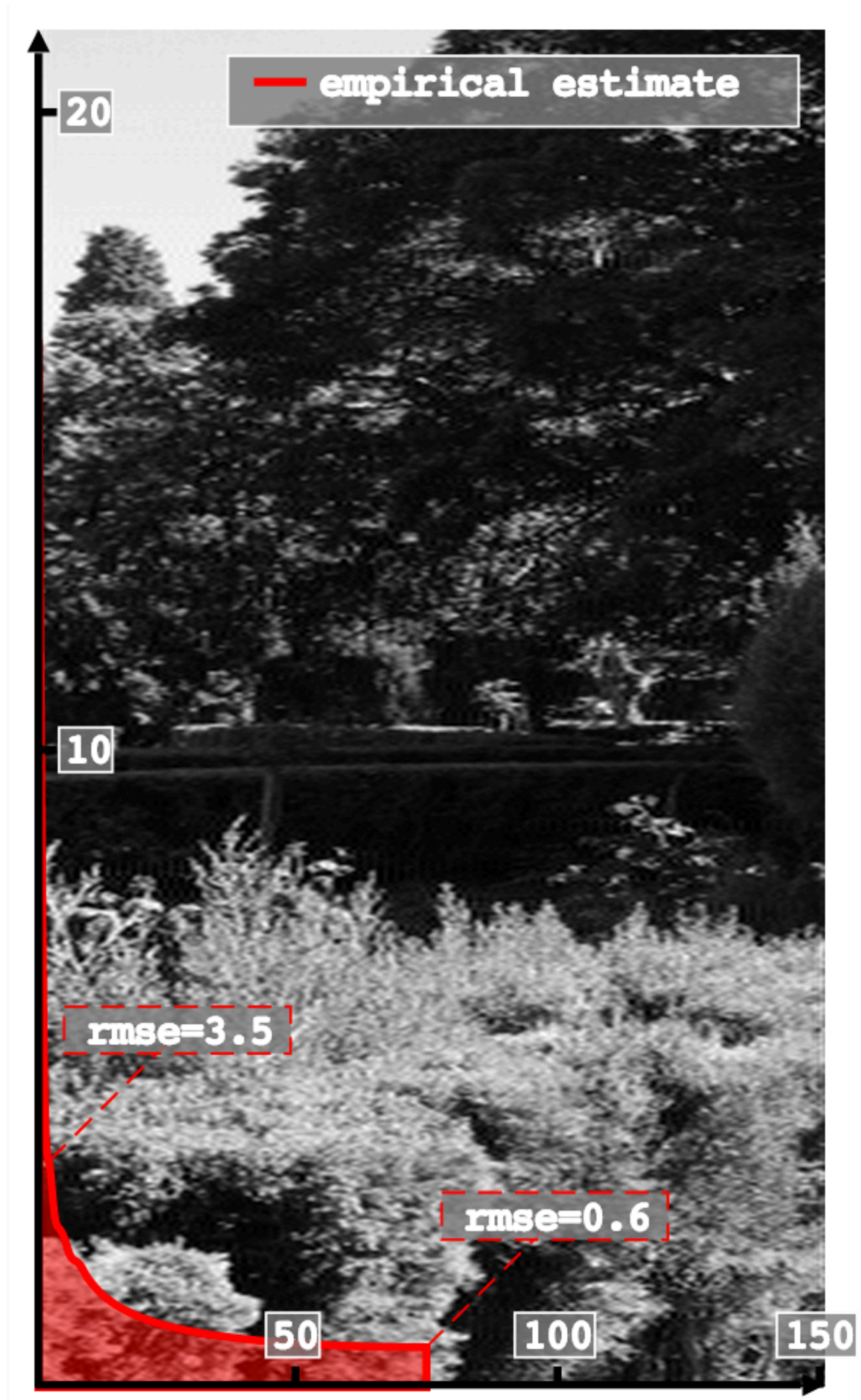
Other kernels?

Sinc w/o μ

Other kernels?



Sinc + accel

Other kernels?

Bilinear

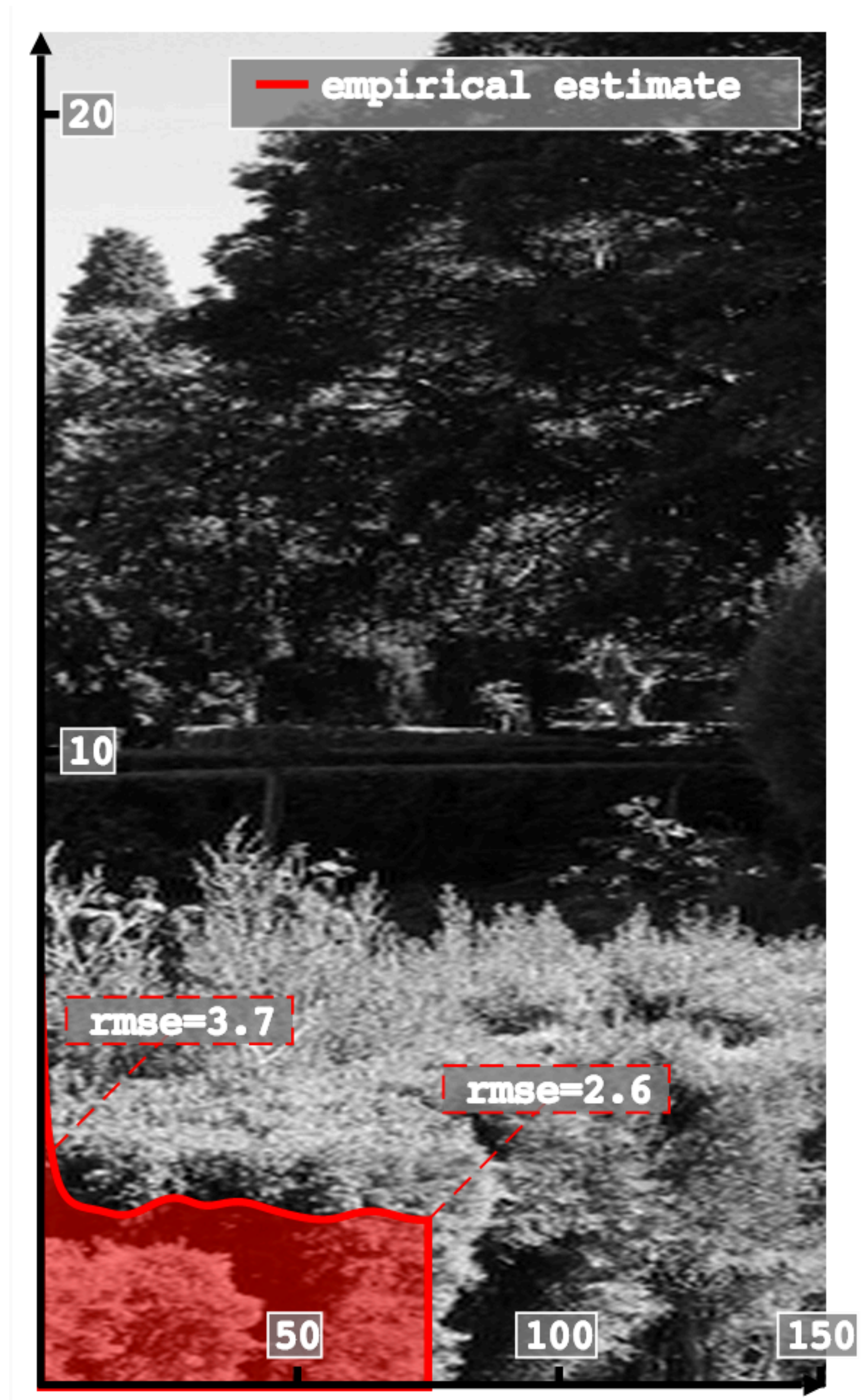
Other kernels?

Bicubic

Other kernels?

B-Spline 3

Other kernels?

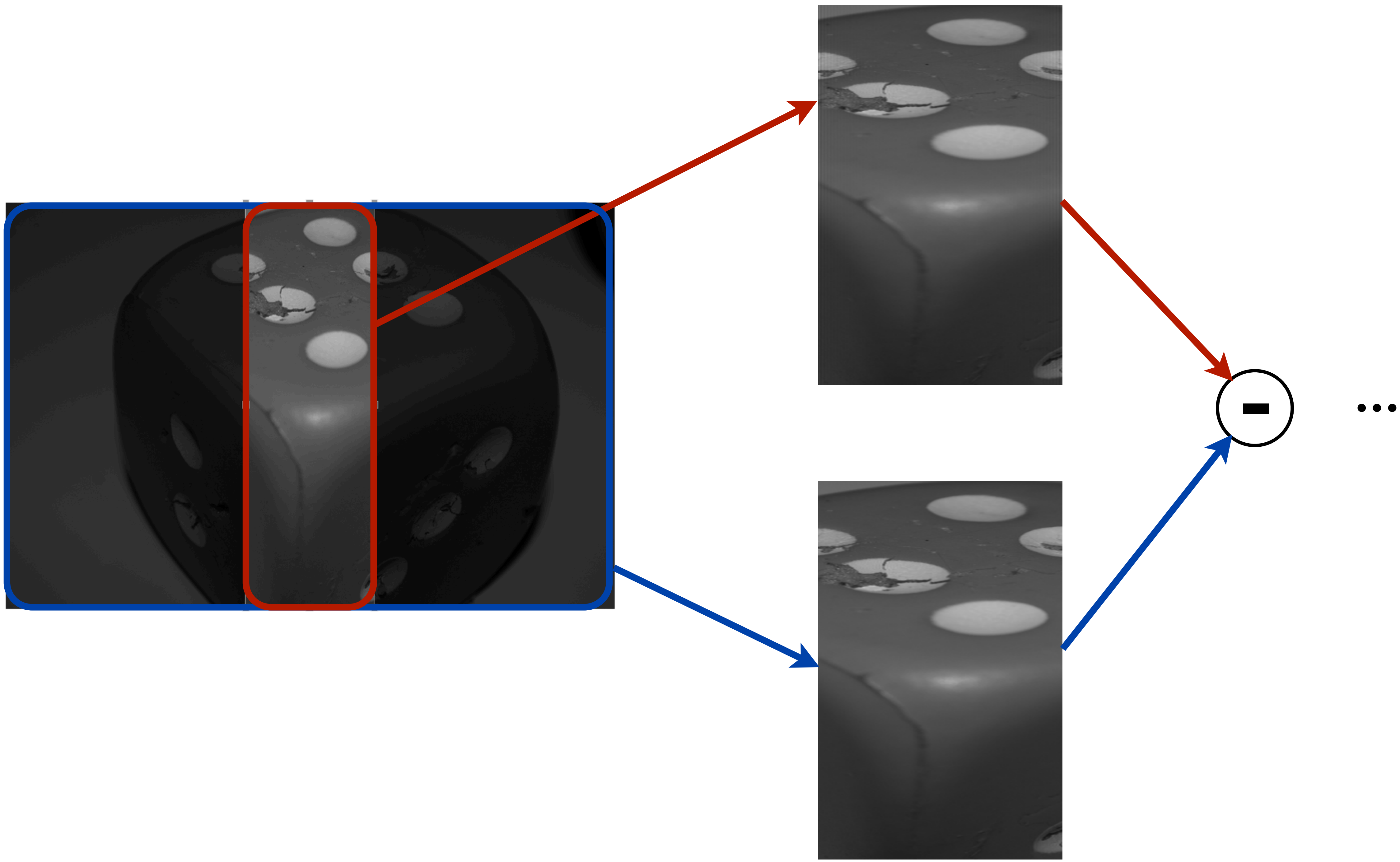


B-Spline 11

Conclusion

- Textures are nasty
- Aliasing is not the worst thing in life
- Is there a hope for image interpolation?

Empirical estimate



Textures

- What's special about images?