

Truncation Error in Image Interpolation

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SampTA 2013 - Bremen



Collaborator

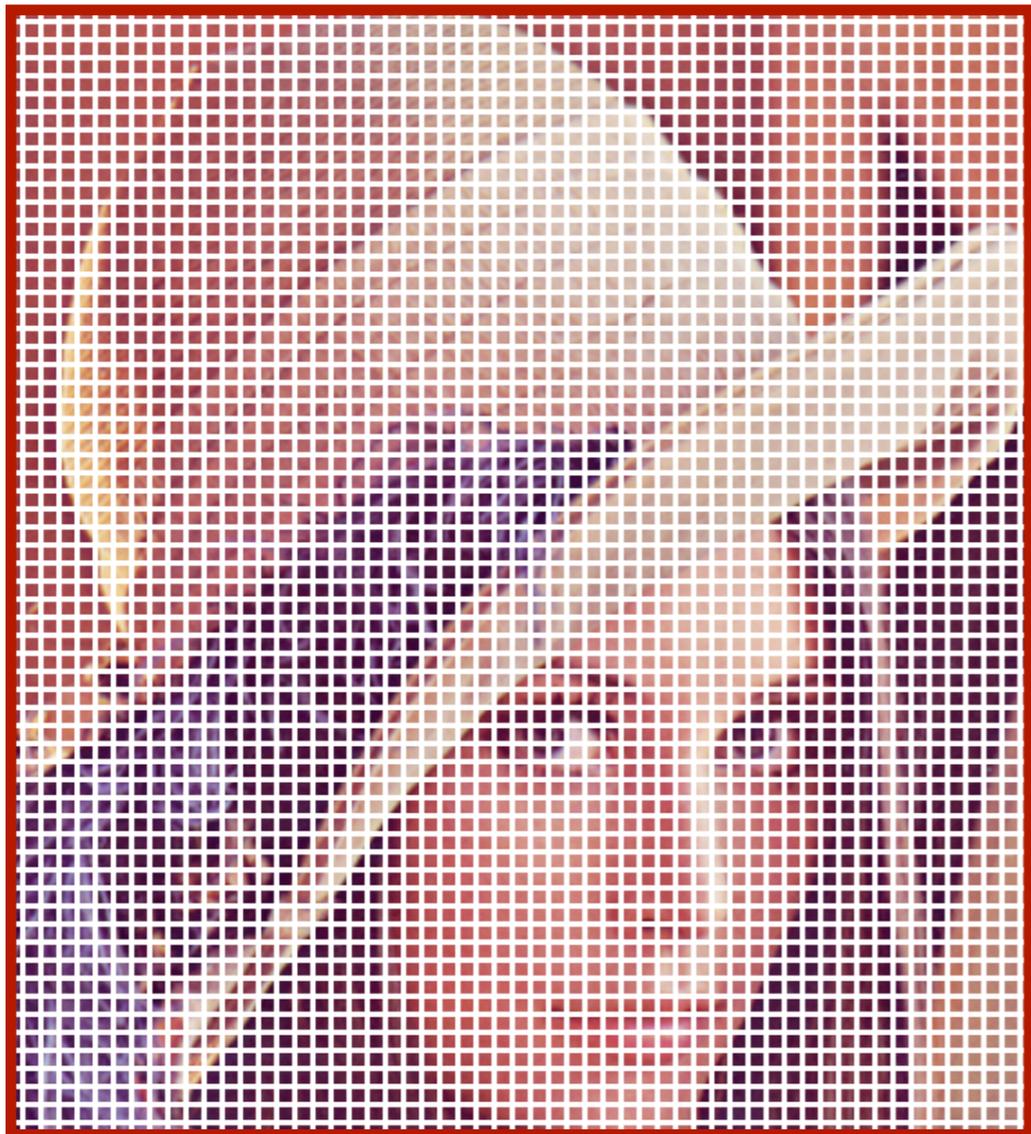


Jean-Michel Morel

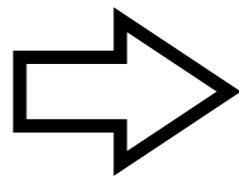


Truncation error:

What is that?



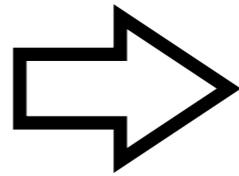
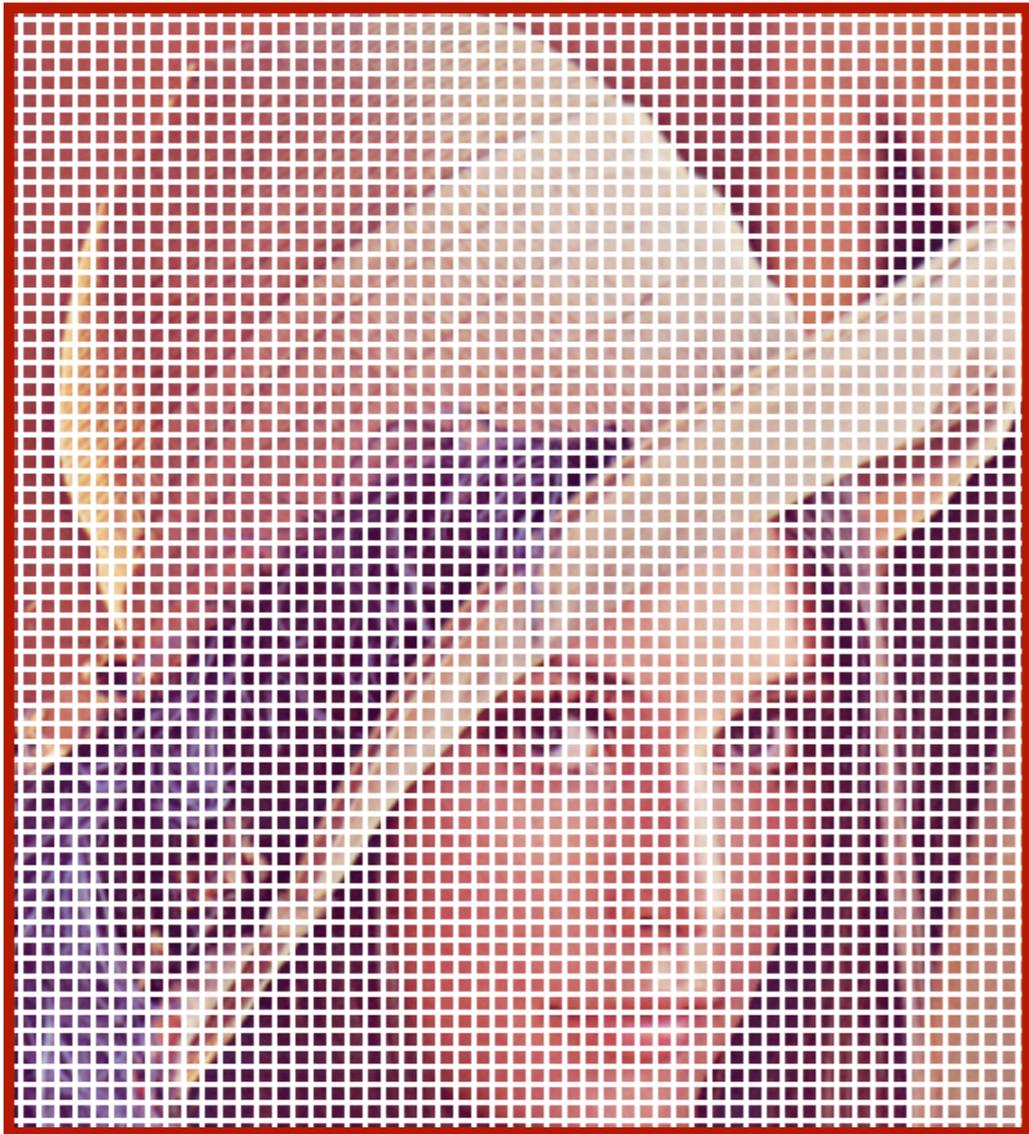
X_k 's



X_t 's

Truncation error:

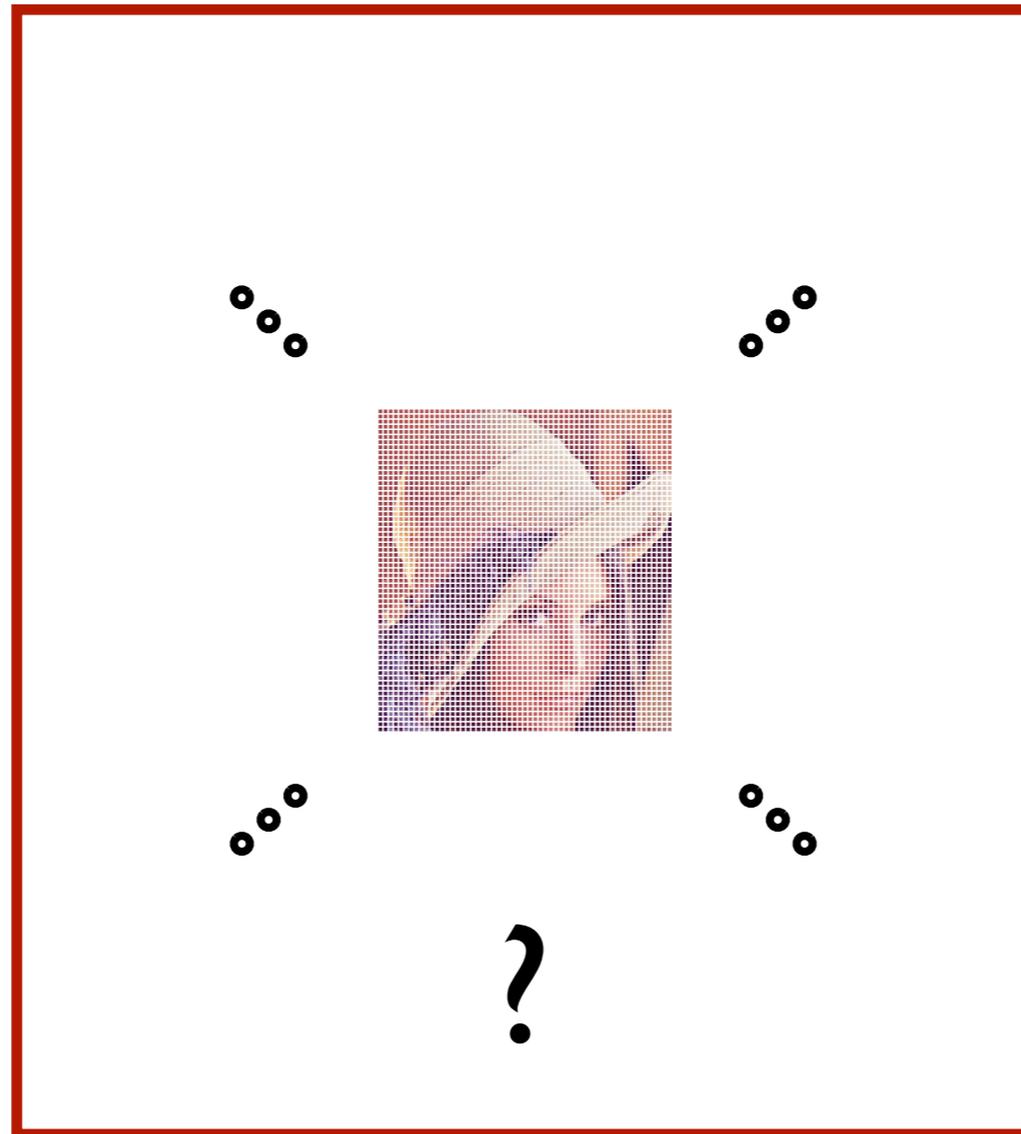
What is that?



$$X_t := \sum_{k \in \mathbb{Z}^2} X_k \text{sinc}(t - k)$$

Truncation error:

What is that?

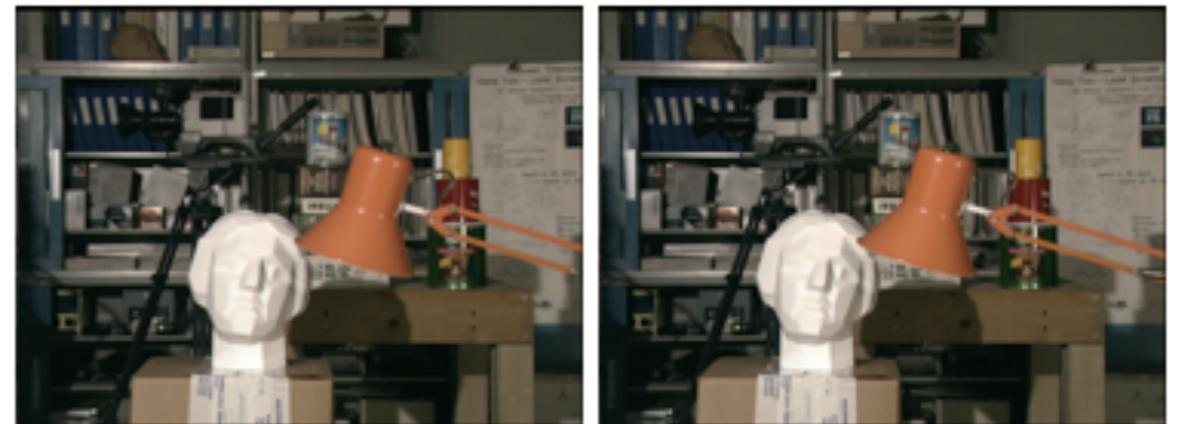


Context

- Motivations
- Assumptions
- Goal
- Related work

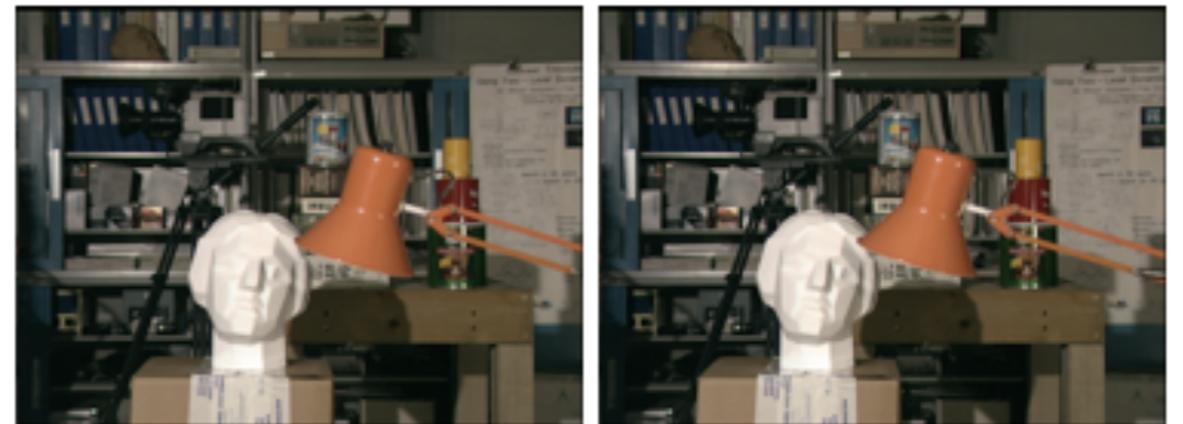
Motivations

- Image registration
- optical flow
- stereopsis
- super-resolution
- sub-pixel accuracy

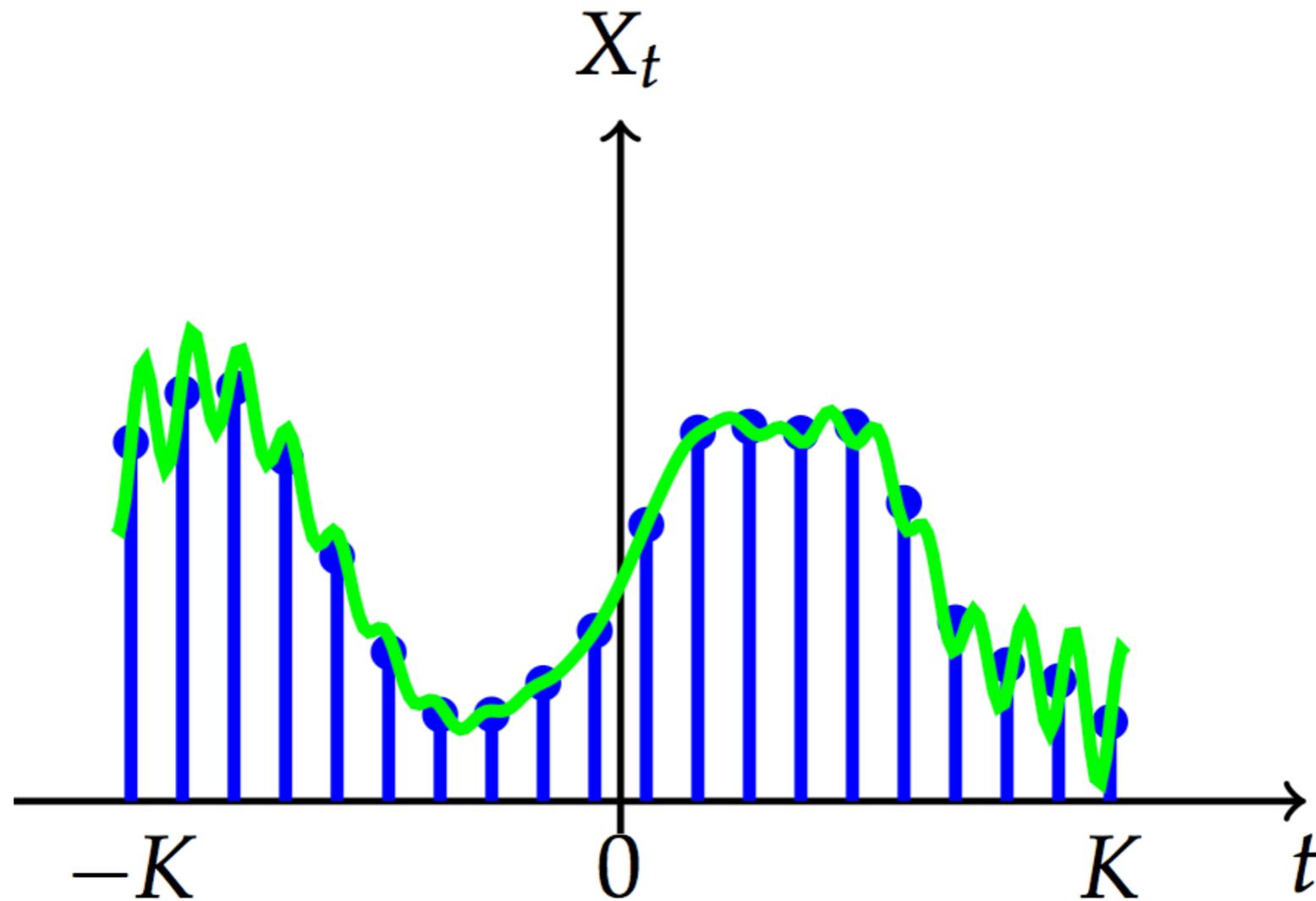


Motivations

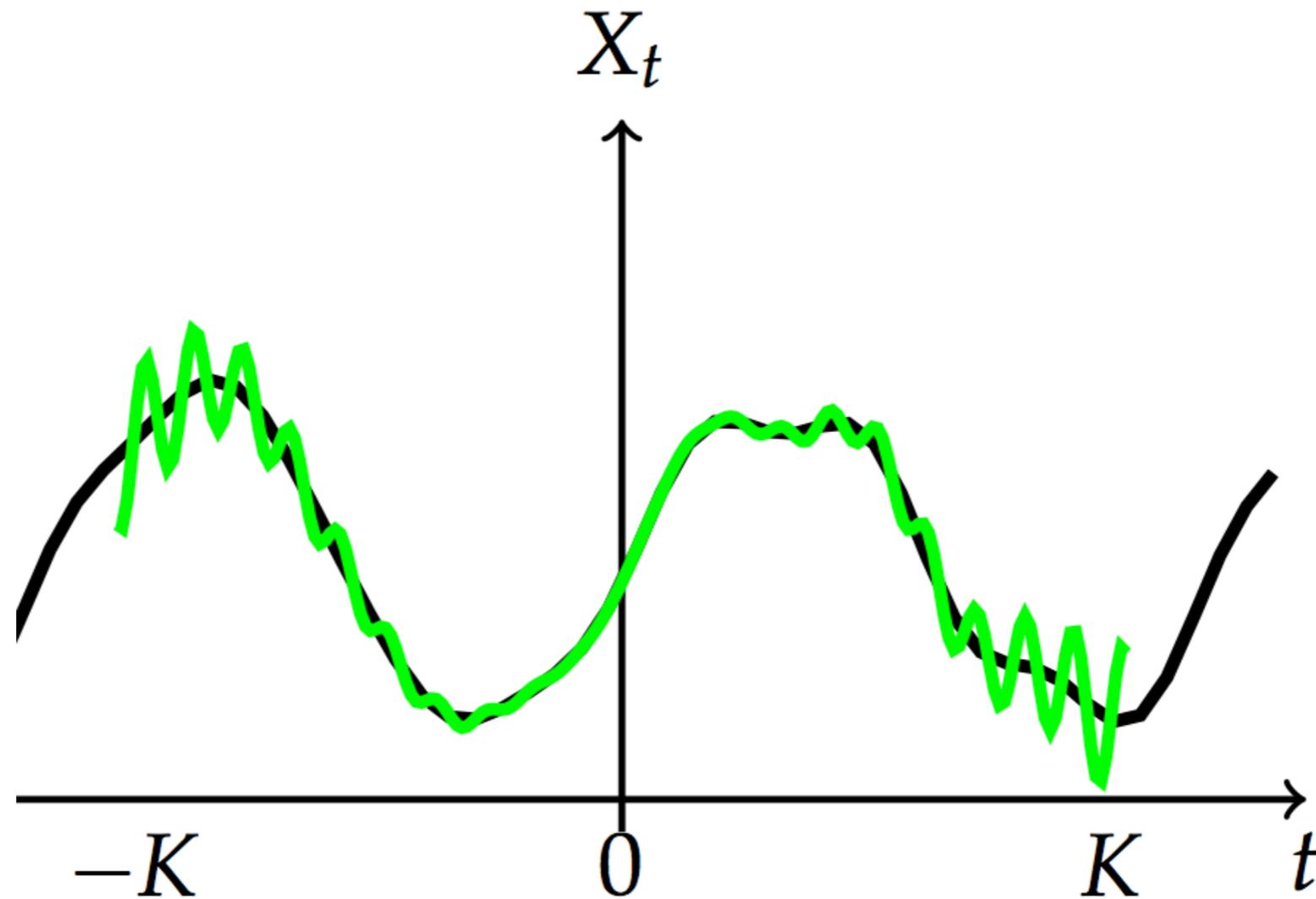
- Image registration
- optical flow
- stereopsis
- super-resolution
- sub-pixel accuracy
- error \sim **quantization**



Assumptions



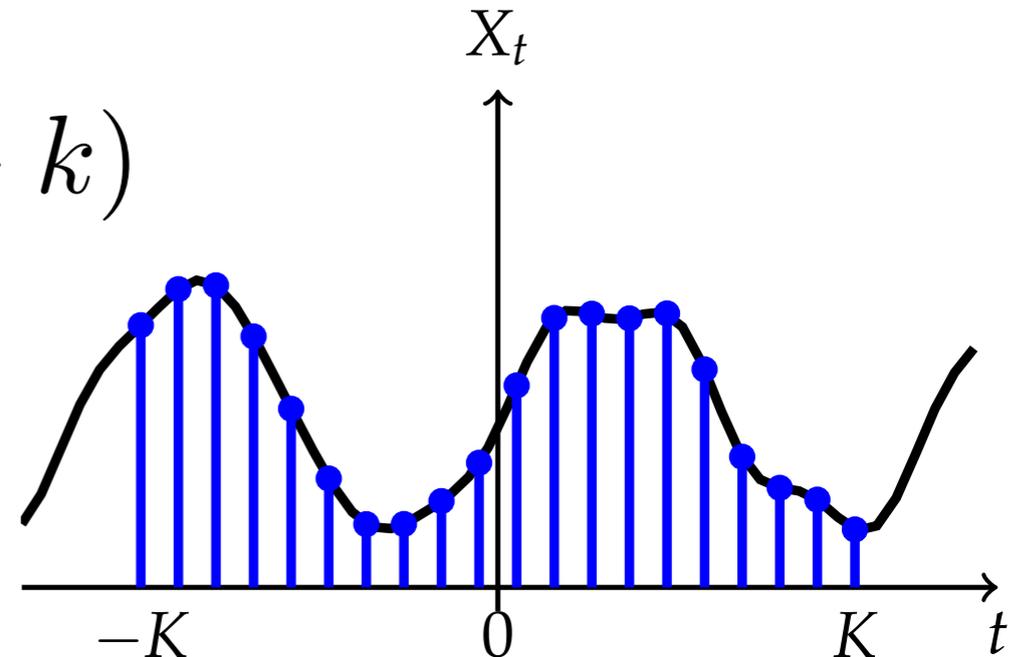
Assumptions



Assumptions

- X_t ($t \in \mathbb{R}$) a 1d random process
 - observed on $k \in \{-K, \dots, K\}$
 - weakly **stationary** $\mu, d\Psi_X(\omega)$
 - **no aliasing**

$$X_t := \sum_{k \in \mathbb{Z}} X_k \text{sinc}(t - k)$$



Goal

- Linear shift-invariant

$$\tilde{X}_t := \sum_{k \leq K} X_k h(t - k)$$

- Practical bounds on

$$RMSE[\tilde{X}_t] := \sqrt{\mathbb{E} \left[(\tilde{X}_t - X_t)^2 \right]}$$

Goal

- Linear shift-invariant

$$\tilde{X}_t := \sum_{k \leq K} X_k h(t - k) \quad \begin{cases} h(t) = \text{sinc}(t) \\ h(t) = \text{sincd}_K(t) \end{cases}$$

- Practical bounds on

$$RMSE[\tilde{X}_t] := \sqrt{\mathbb{E} \left[(\tilde{X}_t - X_t)^2 \right]}$$

➔ Sinc interpolation

➔ DFT interpolation

Related Work

Truncation Error

Jagerman	1966
Yao & Thomas	1966
Campbell	1968
Brown	1969
Xu & Huang & Li	2009

Approximation

Strang & Fix	1971
Blu & Unser	1999
Condat & al.	2005

Other

Jerri	1977	Moisan	2011
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Related Work

Truncation Error

	Jagerman	1966
	Yao & Thomas	1966
	Campbell	1968
	Brown	1969
	Xu & Huang & Li	2009

→ sinc only → oversampled case

Other

Jerri

1977

Moisan

2011

Approximation

Strang & Fix

1971

Blu & Unser

1999

Condat & al.

2005

Related Work

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Blu & Unser	1999
Condat & al.	2005

→ $K = \infty$

Other

Jerri	1977	Moisan	2011
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Related Work

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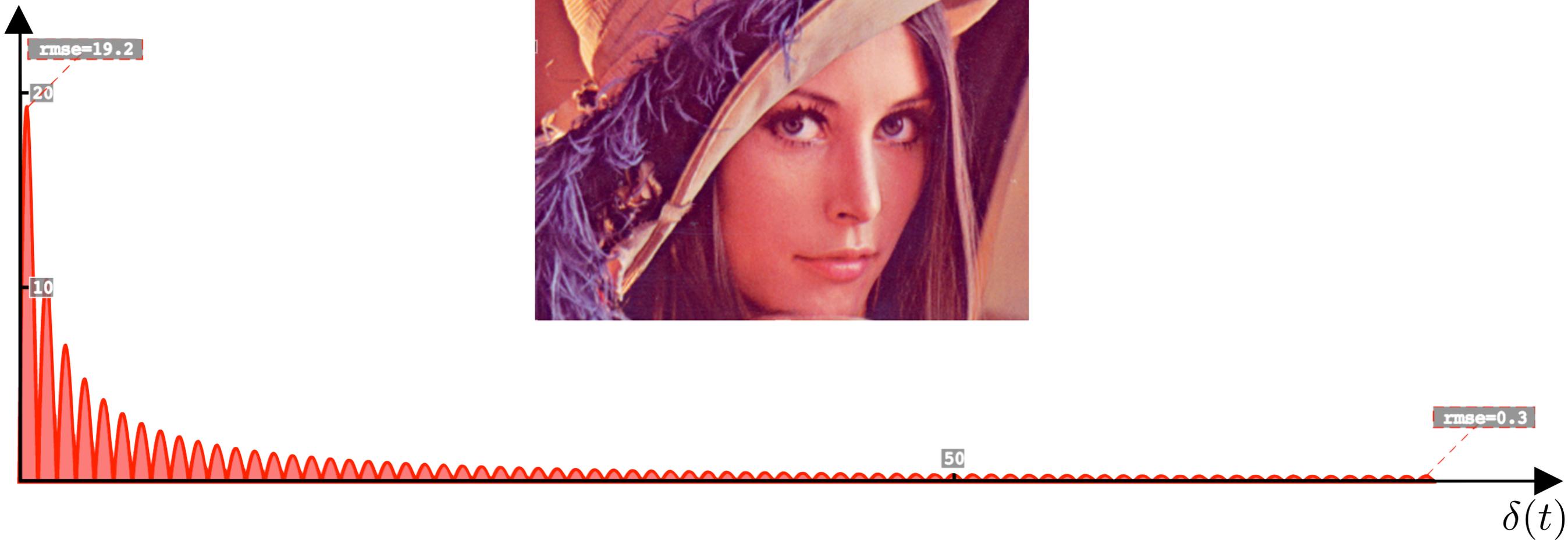
Other

Jerri	1977	Moisan	2011
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Rest of the talk

- Theoretical bounds
- Experimental results
- Discussion & conclusion

A bit of intuition...



Theoretical bounds

$$MSE[\tilde{X}](t) = \frac{\sin^2(\pi t)}{\pi^2} \times \left(\begin{array}{c} \mu^2 \mathcal{O} \left(\frac{1}{\delta(t)^2} \right) \\ + \\ \sigma'_\alpha{}^2 \mathcal{O} \left(\frac{1}{\delta(t)^2} \right) \\ + \\ \sigma_\alpha^2 \mathcal{O} \left(\frac{1}{\delta(t)} \right) \end{array} \right)$$

Theoretical bounds

$$MSE[\tilde{X}](t) = \frac{\sin^2(\pi t)}{\pi^2} \times \left(\begin{array}{c} \mu^2 \mathcal{O} \left(\frac{1}{\delta(t)^2} \right) \\ + \\ \sigma_{\alpha}'^2 \mathcal{O} \left(\frac{1}{\delta(t)^2} \right) \\ + \\ \sigma_{\alpha}^2 \mathcal{O} \left(\frac{1}{\delta(t)} \right) \end{array} \right)$$

Theoretical bounds

$$MSE[\tilde{X}](t) = \frac{\sin^2(\pi t)}{\pi^2} \times \left(\begin{array}{c} 0\mu^2 \mathcal{O}\left(\frac{1}{\delta(t)^2}\right) \\ + \\ 2\sigma'_\alpha{}^2 \mathcal{O}\left(\frac{1}{\delta(t)^2}\right) \\ + \\ 2\sigma_\alpha^2 \mathcal{O}\left(\frac{1}{\delta(t)}\right) \end{array} \right)$$

➔ DFT modifications

Spectral representation

$$MSE[\tilde{X}](t) = \underbrace{\mu^2 \left| 1 - \sum_{|k| \leq K} h(t-k) \right|^2}_{MSE[\mu](t)} + \underbrace{\frac{1}{2\pi} \int \left| e^{i\omega t} - \sum_{|k| \leq K} e^{i\omega k} h(t-k) \right|^2 d\Psi_X(\omega)}_{MSE[d\Psi_X](t)}$$

➔ Average component ➔ Spectral component

Spectral representation

$$\begin{aligned} MSE[\tilde{X}](t) &= \underbrace{\mu^2 \left| 1 - \sum_{|k| \leq K} h(t-k) \right|^2}_{MSE[\mu](t)} \\ &+ \\ &\underbrace{\frac{1}{2\pi} \int \left| e^{i\omega t} - \sum_{|k| \leq K} e^{i\omega k} h(t-k) \right|^2 d\Psi_X(\omega)}_{MSE[d\Psi_X](t)} \end{aligned}$$

➔ Aliasing is not forbidden

Spectral representation

$$\begin{aligned}
 MSE[\tilde{X}](t) &= \underbrace{\mu^2 \left| \sum_{k \in \mathbb{Z}} \text{sinc}(t - k) - \sum_{|k| \leq K} h(t - k) \right|^2}_{MSE[\mu](t)} \\
 &+ \\
 &\underbrace{\frac{1}{2\pi} \int_{|\omega| \leq \pi} \left| \sum_{k \in \mathbb{Z}} e^{i\omega k} \text{sinc}(t - k) - \sum_{|k| \leq K} e^{i\omega k} h(t - k) \right|^2 d\Psi_X(\omega)}_{MSE[d\Psi_X](t)}
 \end{aligned}$$

➔ Under no aliasing condition

Spectral representation

$$\begin{aligned}
 MSE[\tilde{X}](t) = & \underbrace{\mu^2 \left| \sum_{k \in \mathbb{Z}} \text{sinc}(t - k) - \sum_{|k| \leq K} \text{sinc}(t - k) \right|^2}_{MSE[\mu](t)} \\
 & + \\
 & \underbrace{\frac{1}{2\pi} \int_{|\omega| \leq \pi} \left| \sum_{k \in \mathbb{Z}} e^{i\omega k} \text{sinc}(t - k) - \sum_{|k| \leq K} e^{i\omega k} \text{sinc}(t - k) \right|^2 d\Psi_X(\omega)}_{MSE[d\Psi_X](t)}
 \end{aligned}$$

➔ Under no aliasing condition

➔ Sinc

Average component

$$MSE[\mu](t) = \frac{\sin^2(\pi t)}{\pi^2} \mu^2 \mathcal{O}\left(\frac{1}{\delta(t)^2}\right)$$

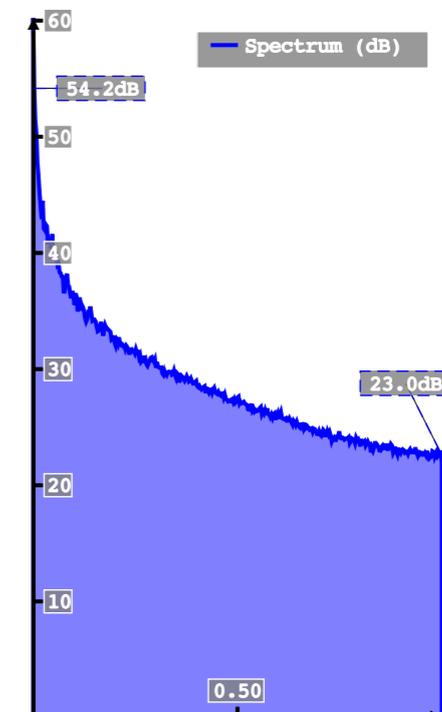
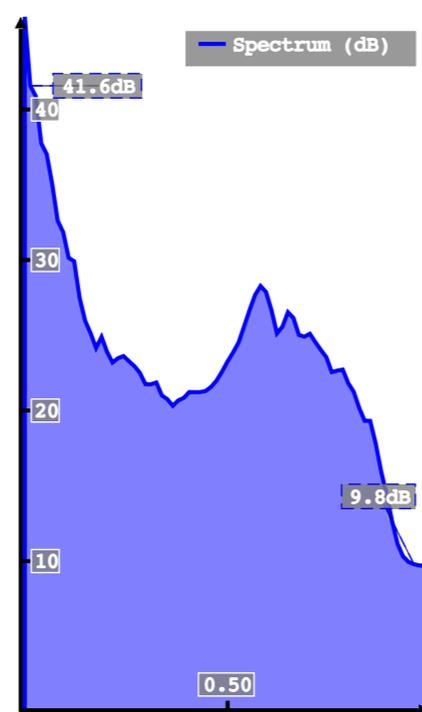
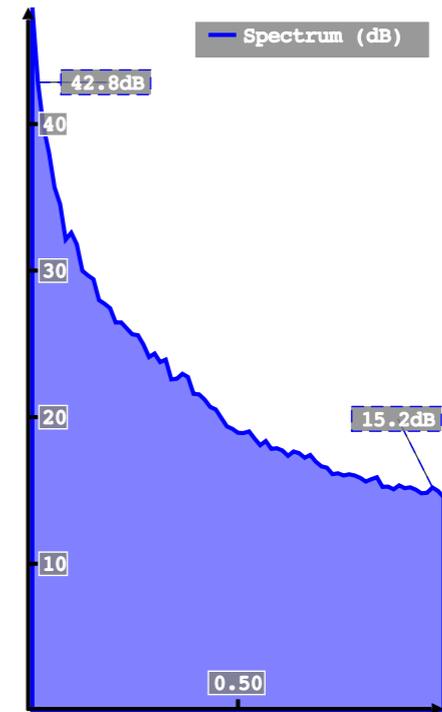
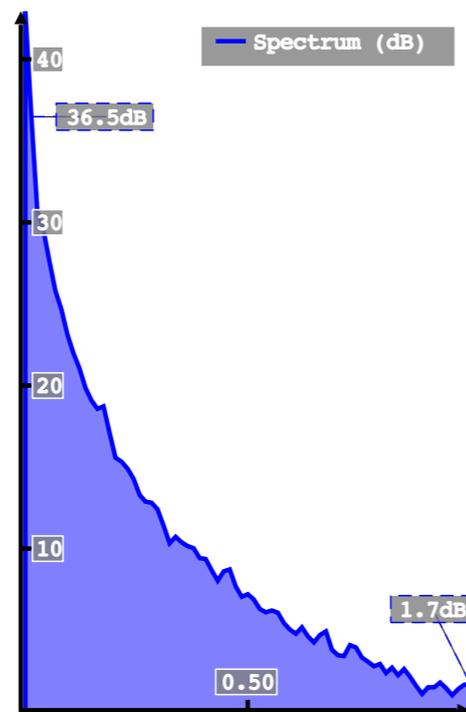
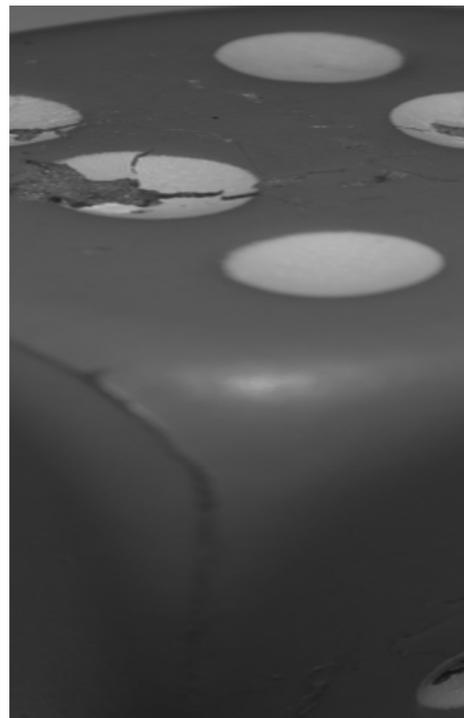
➔ Gibbs phenomenon

Average component

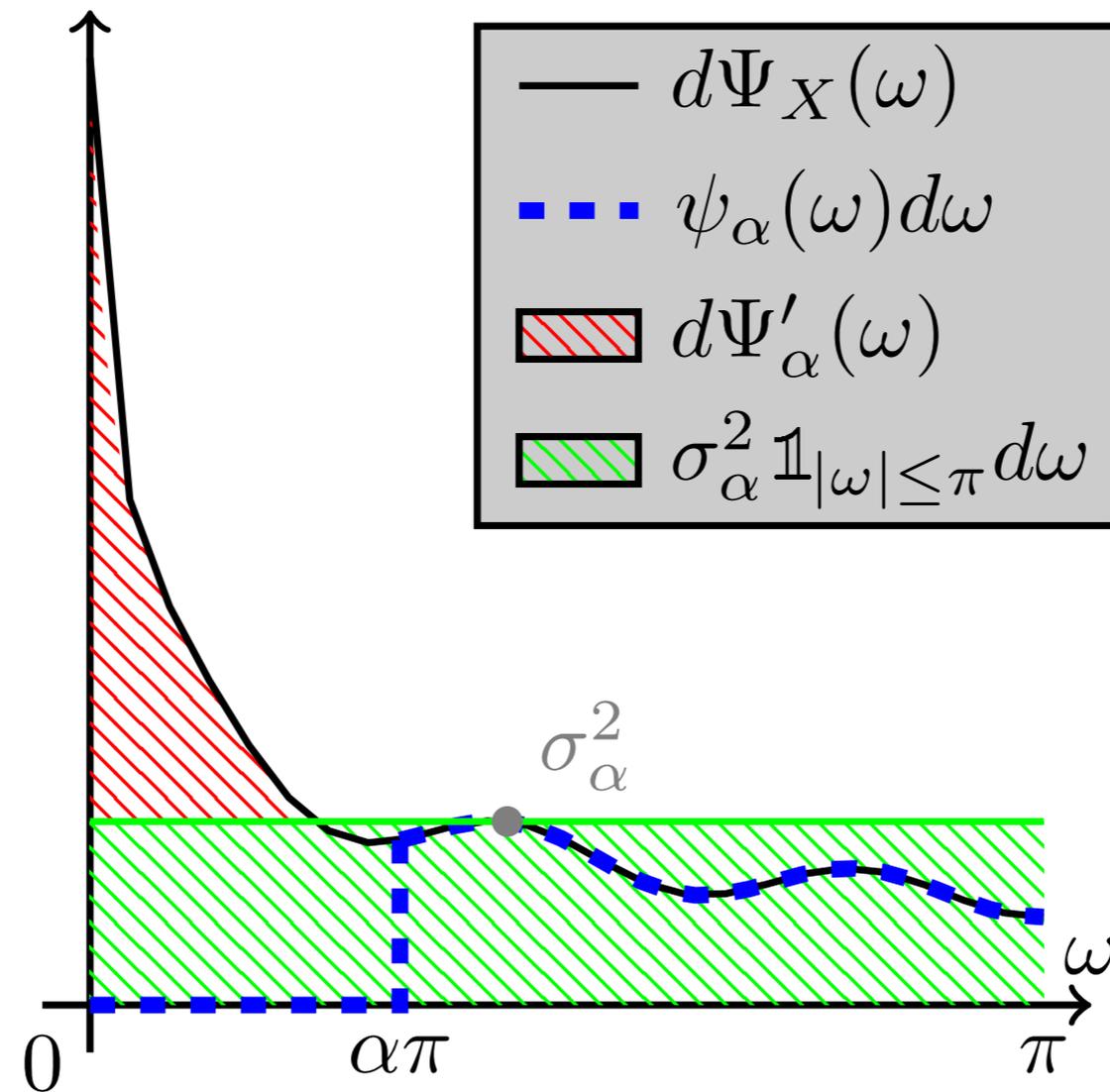
$$MSE[\mu](t) = 0$$

➔ DFT

Spectral component



Spectral decomposition



➔ spectrum \leq **oversampled** + **white-noise**

Oversampled case

$$\text{supp}(d\Psi'_\alpha) \subset \{|\omega| \leq \alpha\pi\}$$

\implies

$$MSE[d\Psi'_\alpha](t) = \frac{\sin^2(\pi t)}{\pi^2} \sigma_\alpha'^2 \mathcal{O}\left(\frac{1}{\delta(t)^2}\right)$$

where,

$$\sigma_\alpha'^2 = \frac{1}{\pi} \int_{|\omega| \leq \alpha\pi} \frac{1}{1 + \cos(\omega)} d\Psi'_\alpha(\omega)$$

Oversampled case

$$\text{supp}(d\Psi'_\alpha) \subset \{|\omega| \leq \alpha\pi\}$$

\implies

$$\left\{ \begin{array}{l} \text{MSE}[d\Psi'_\alpha](t) \\ \text{MSE}[\mu](t) \end{array} \right\} = \frac{\sin^2(\pi t)}{\pi^2} \left\{ \begin{array}{l} \sigma_\alpha'^2 \\ \mu^2 \end{array} \right\} \mathcal{O} \left(\frac{1}{\delta(t)^2} \right)$$

where,

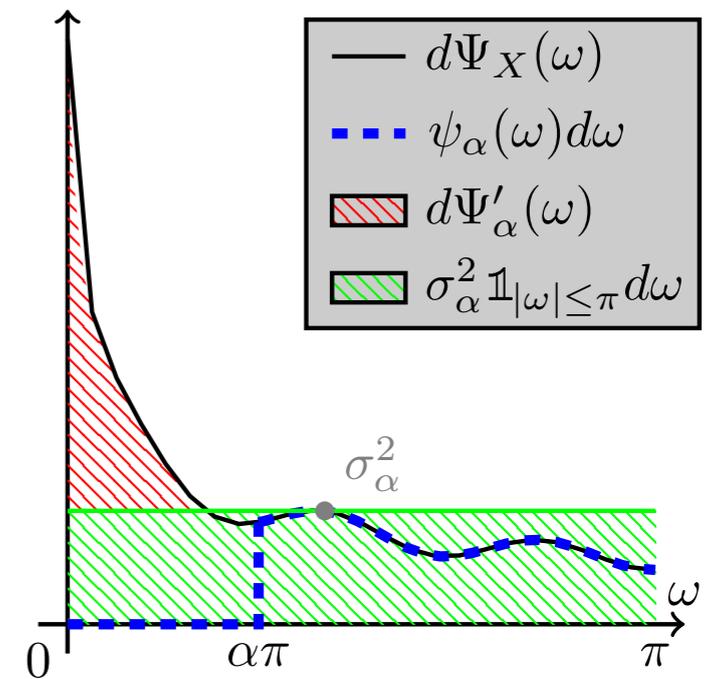
$$\sigma_\alpha'^2 = \frac{1}{\pi} \int_{|\omega| \leq \alpha\pi} \frac{1}{1 + \cos(\omega)} d\Psi'_\alpha(\omega)$$

White-noise

$$d\Psi(\omega) = \sigma_\alpha^2 \mathbb{1}_{|\omega| \leq \pi} d\omega$$

$$\implies$$

$$MSE[d\Psi](t) = \frac{\sin^2(\pi t)}{\pi^2} \sigma_\alpha^2 \mathcal{O}\left(\frac{1}{\delta(t)}\right)$$



➔ Slow decay

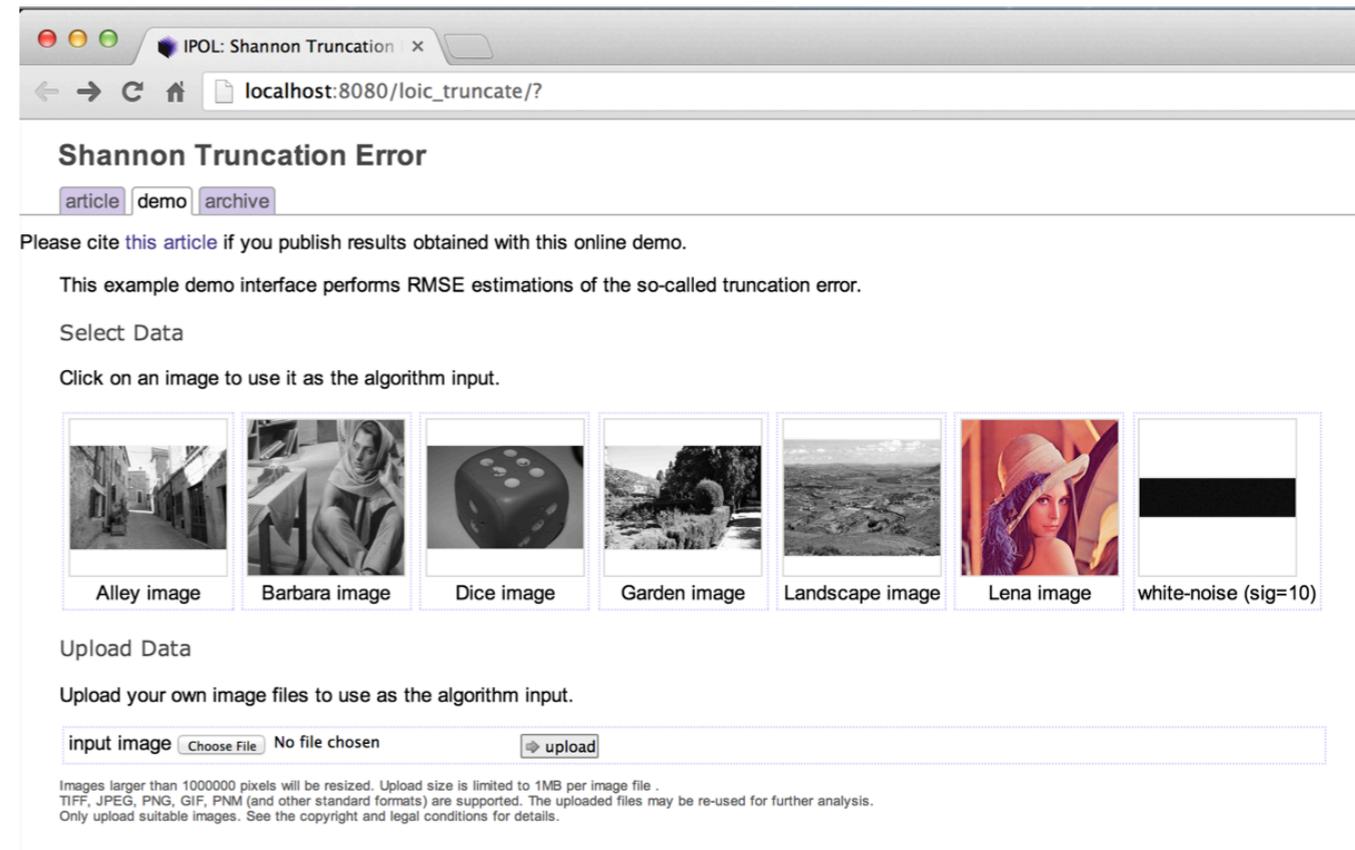
Recap

$$MSE[\tilde{X}](t) = \frac{\sin^2(\pi t)}{\pi^2} \times \left(\begin{array}{c} 0\mu^2 \mathcal{O}\left(\frac{1}{\delta(t)^2}\right) \\ + \\ 2\sigma'_\alpha{}^2 \mathcal{O}\left(\frac{1}{\delta(t)^2}\right) \\ + \\ 2\sigma_\alpha^2 \mathcal{O}\left(\frac{1}{\delta(t)}\right) \end{array} \right)$$

➔ DFT modifications

Experimental results

- Bound validity
- Bound tightness
- Order of magnitude

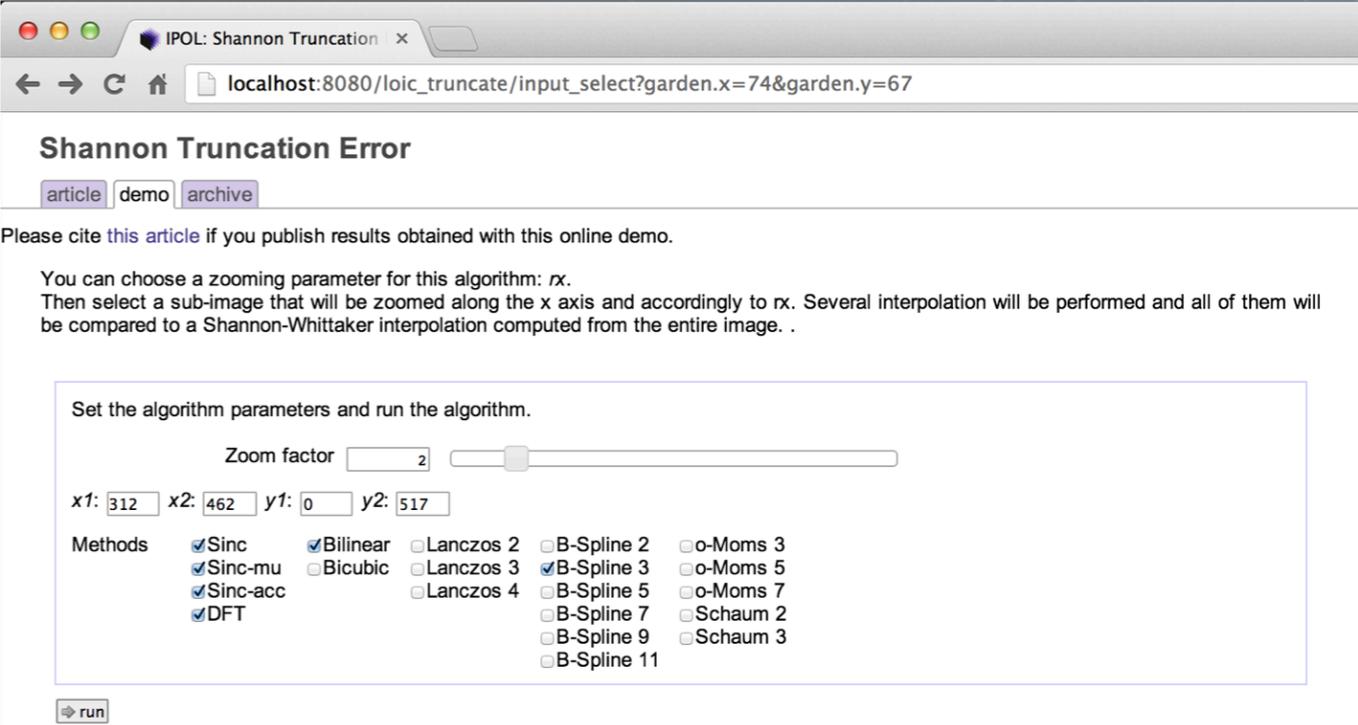


➔ online demo: 

IPOL · Image Processing On Line

Experimental results

- Bound validity
- Bound tightness
- Order of magnitude



The screenshot shows a web browser window titled "IPOL: Shannon Truncation Error". The address bar shows the URL: localhost:8080/loic_truncate/input_select?garden.x=74&garden.y=67. The page content includes a title "Shannon Truncation Error" with links for "article", "demo", and "archive". Below the title, there is a note: "Please cite this article if you publish results obtained with this online demo." and a paragraph explaining the algorithm: "You can choose a zooming parameter for this algorithm: α . Then select a sub-image that will be zoomed along the x axis and accordingly to α . Several interpolation will be performed and all of them will be compared to a Shannon-Whittaker interpolation computed from the entire image. .". The main interface is a form titled "Set the algorithm parameters and run the algorithm." with a "Zoom factor" slider set to 2, and input fields for "x1: 312", "x2: 462", "y1: 0", and "y2: 517". Under "Methods", there are several checkboxes: Sinc, Sinc-mu, Sinc-acc, DFT, Bilinear, Bicubic, Lanczos 2, Lanczos 3, Lanczos 4, B-Spline 2, B-Spline 3, B-Spline 5, B-Spline 7, B-Spline 9, B-Spline 11, o-Moms 3, o-Moms 5, o-Moms 7, Schaum 2, and Schaum 3. A "run" button is located at the bottom left of the form.

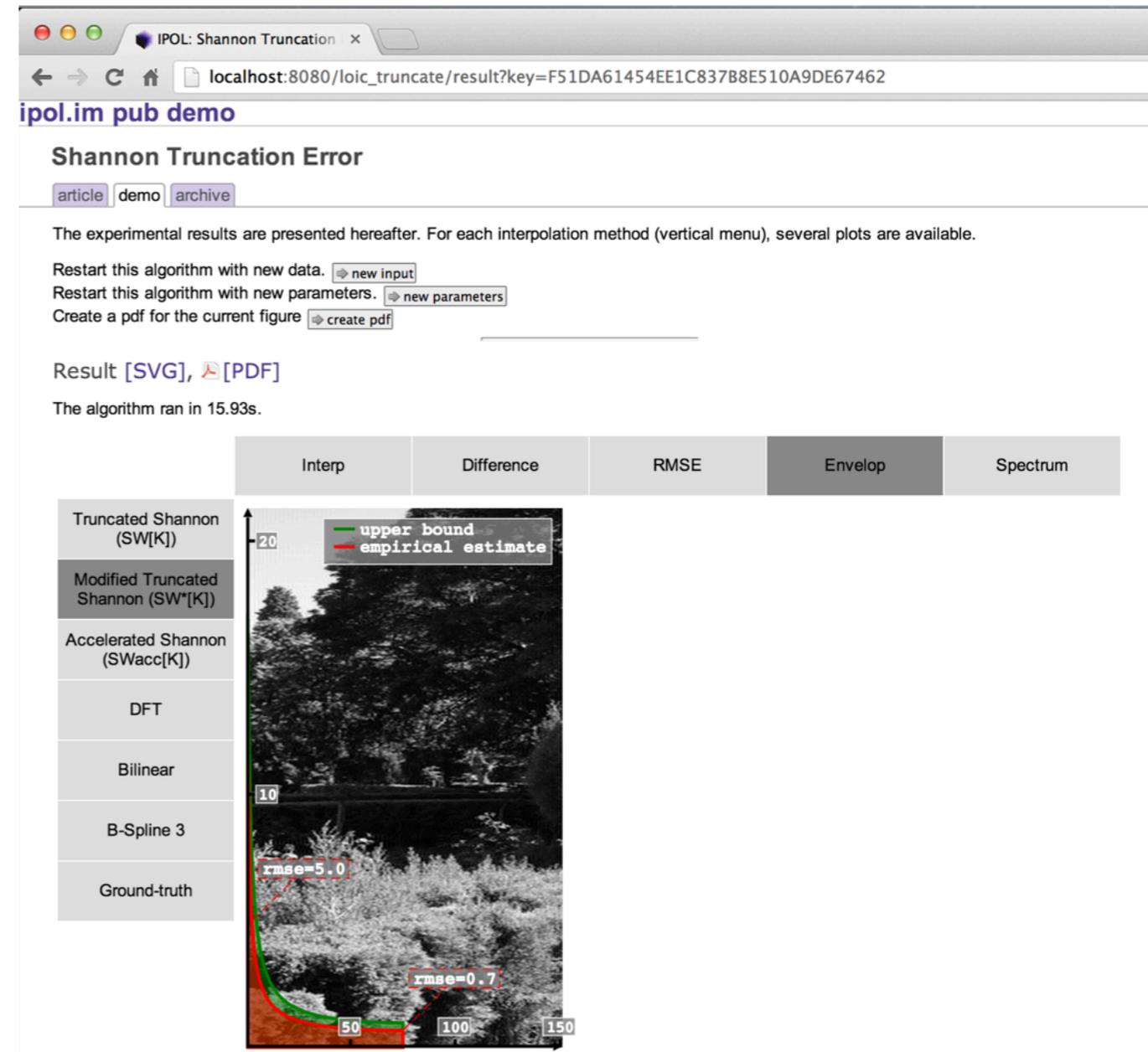
➔ online demo:



IPOL · Image Processing On Line

Experimental results

- Bound validity
- Bound tightness
- Order of magnitude



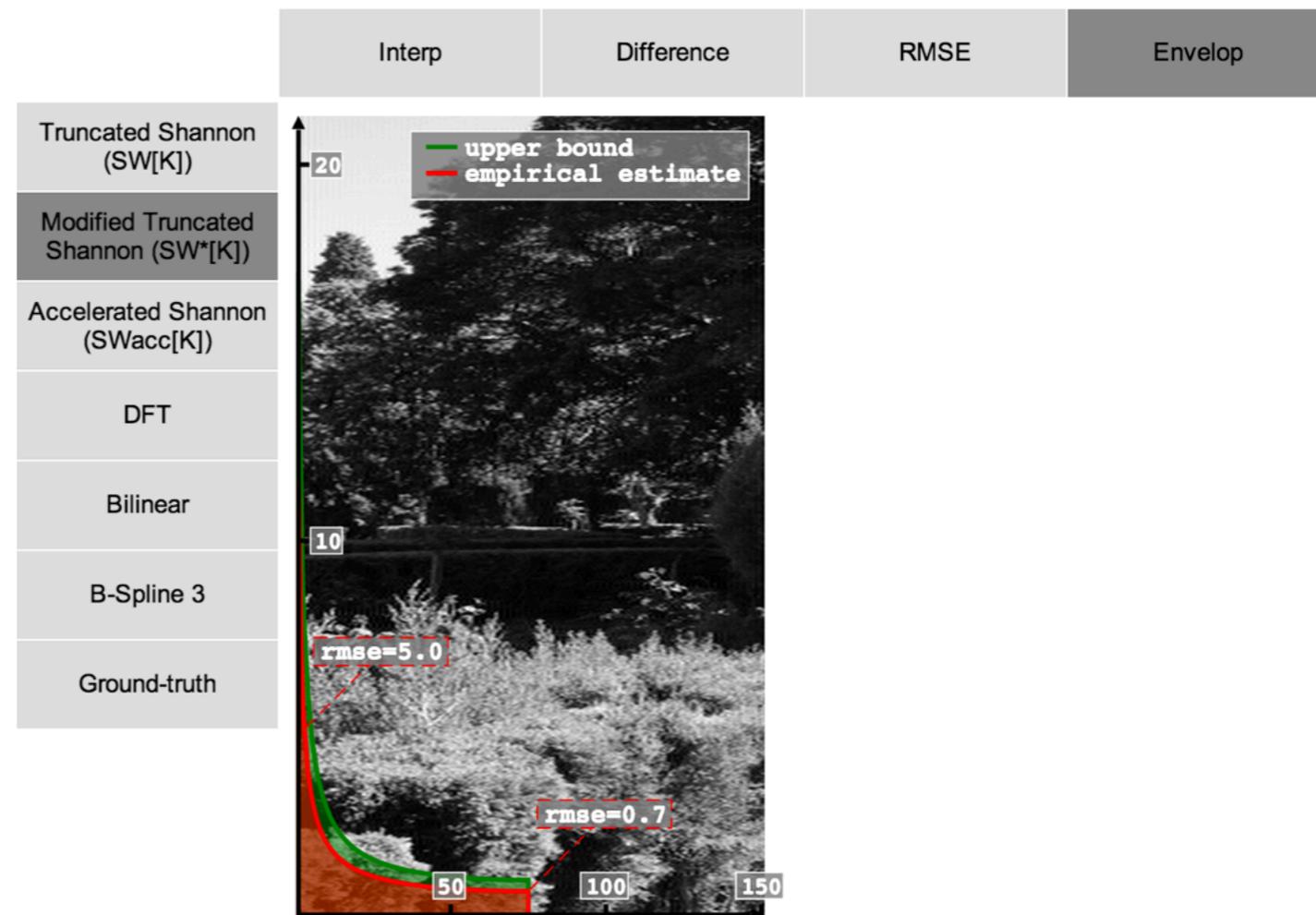
➔ online demo:



IPOL · Image Processing On Line

Experimental results

- Bound validity
- Bound tightness
- Order of magnitude

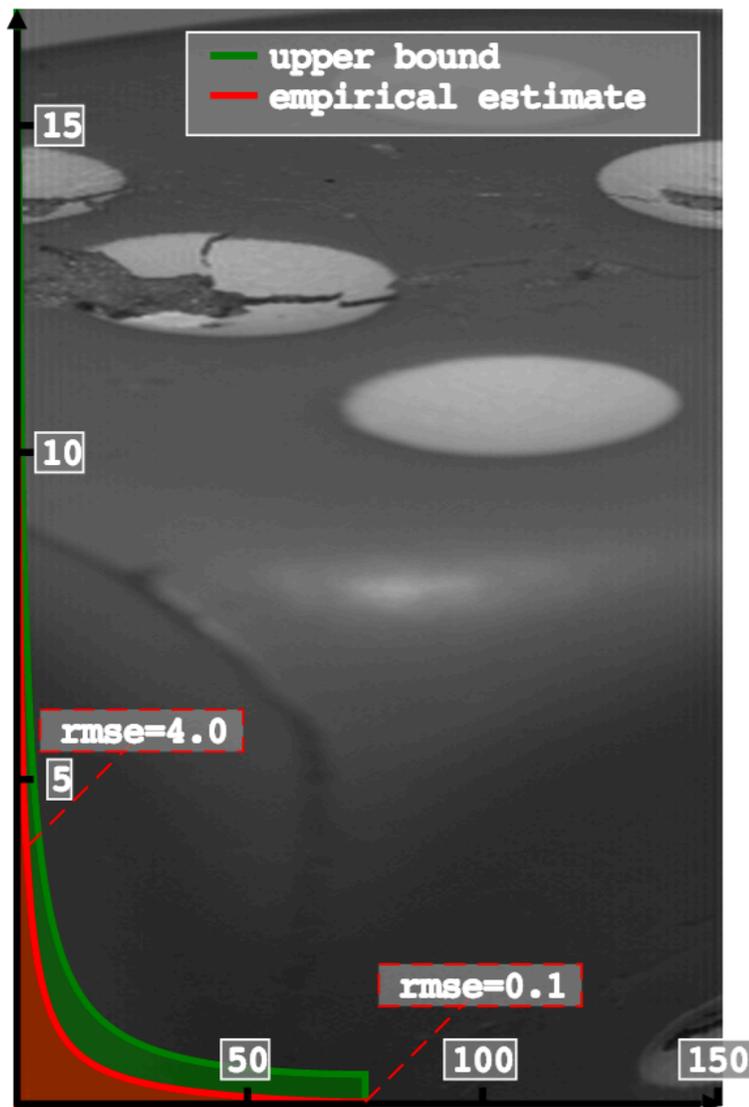


➔ online demo:



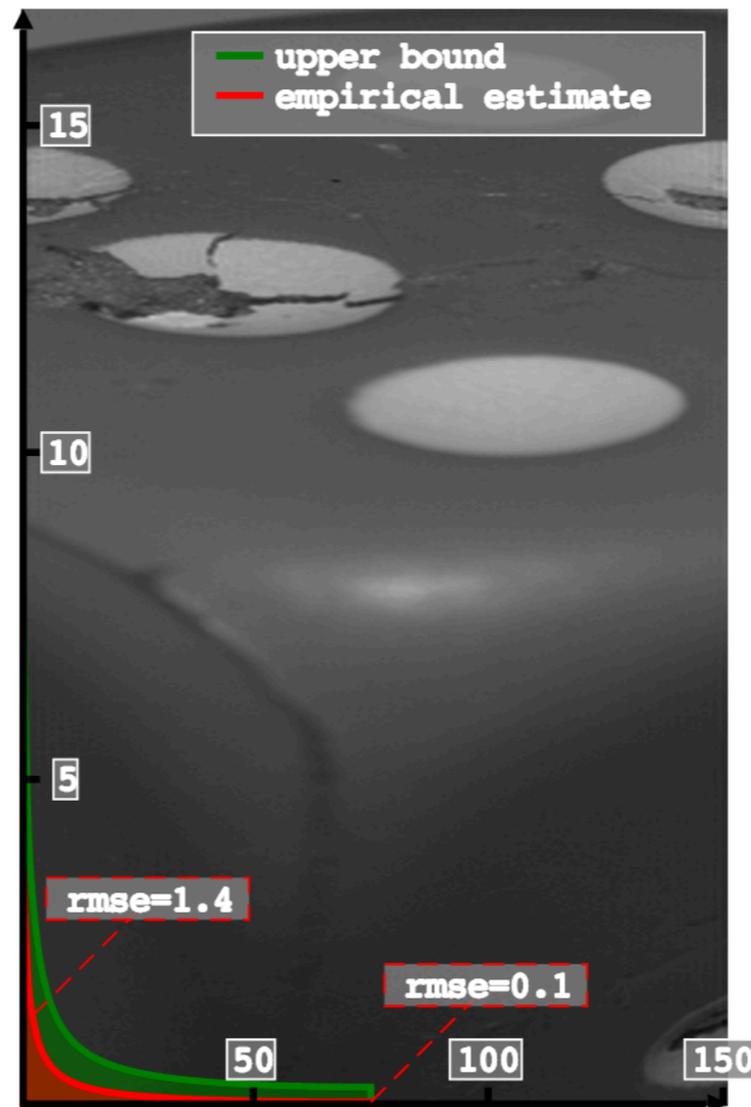
IPOL · Image Processing On Line

Bound tightness

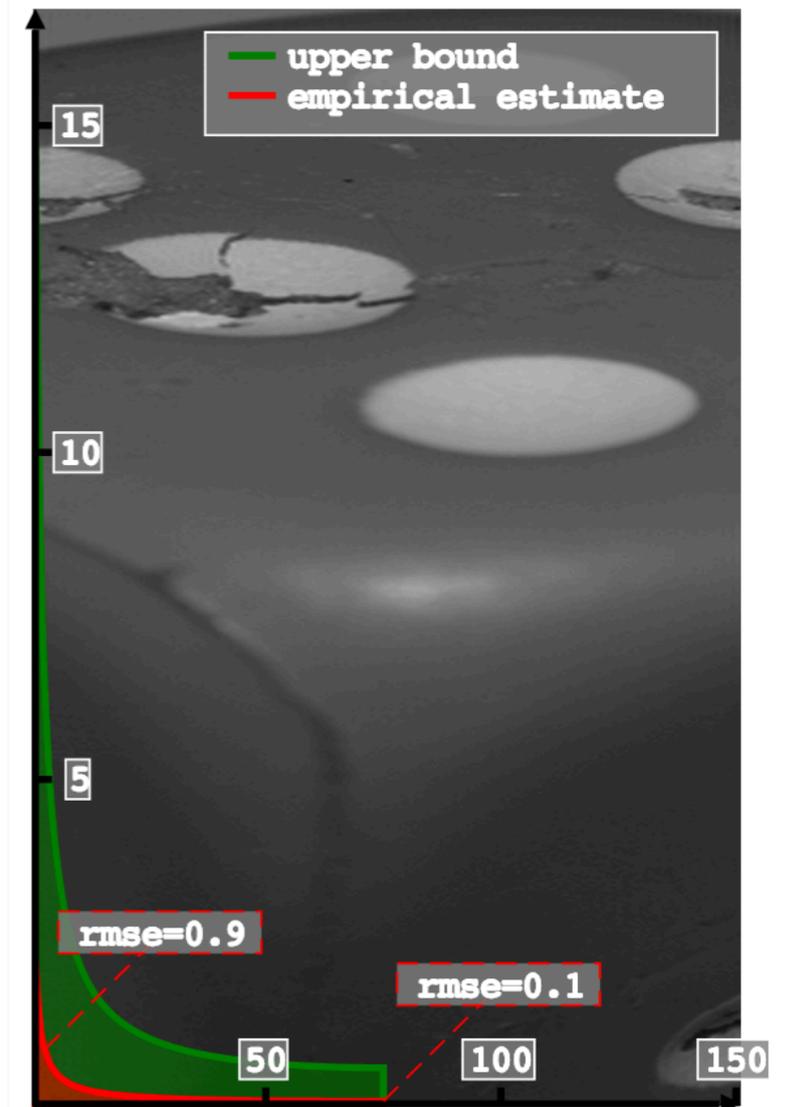


Sinc

➔ Smooth image



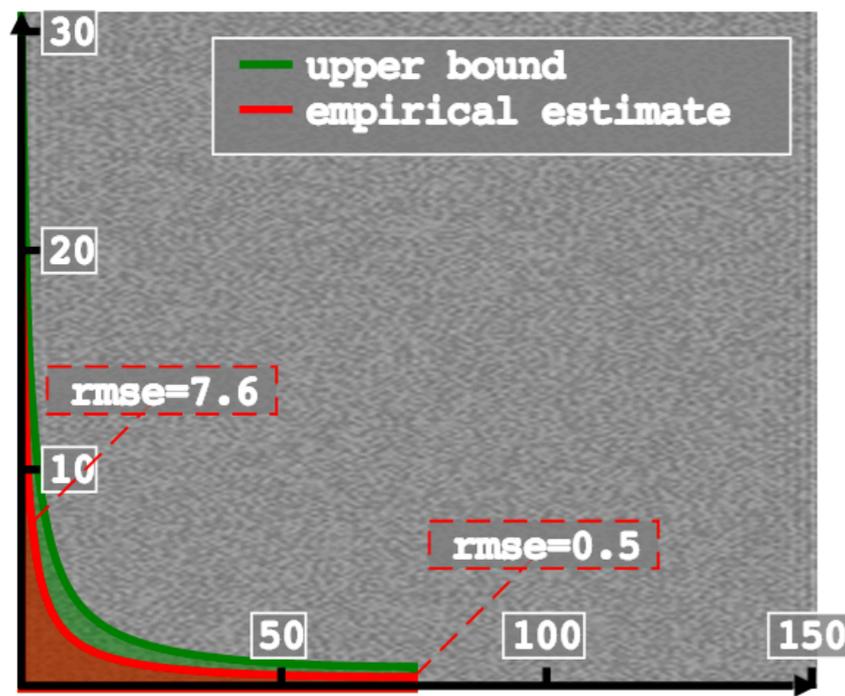
Sinc w/o μ



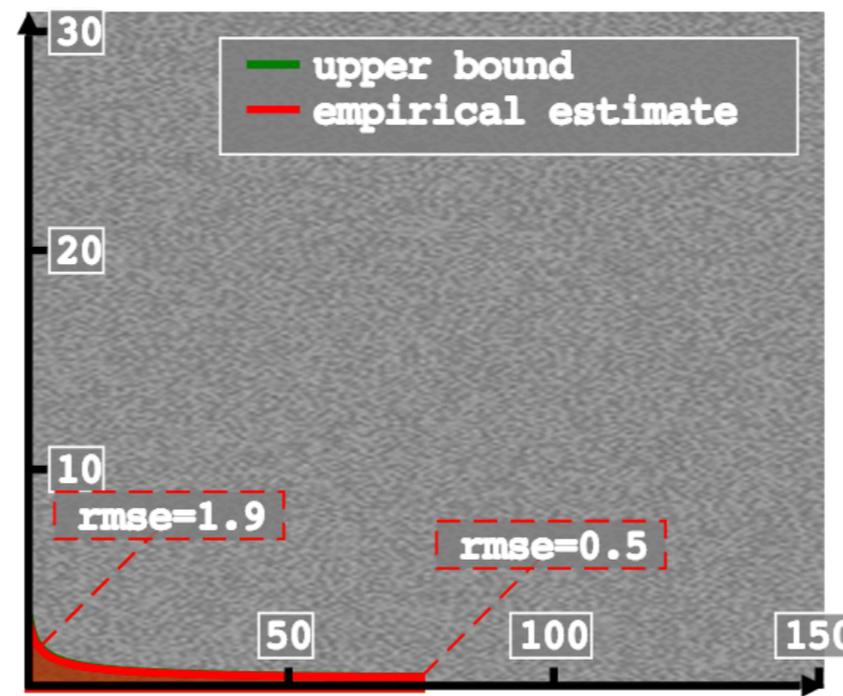
DFT

➔ $\sqrt{\mathbb{E}[\text{quant}^2]} = 0.3$

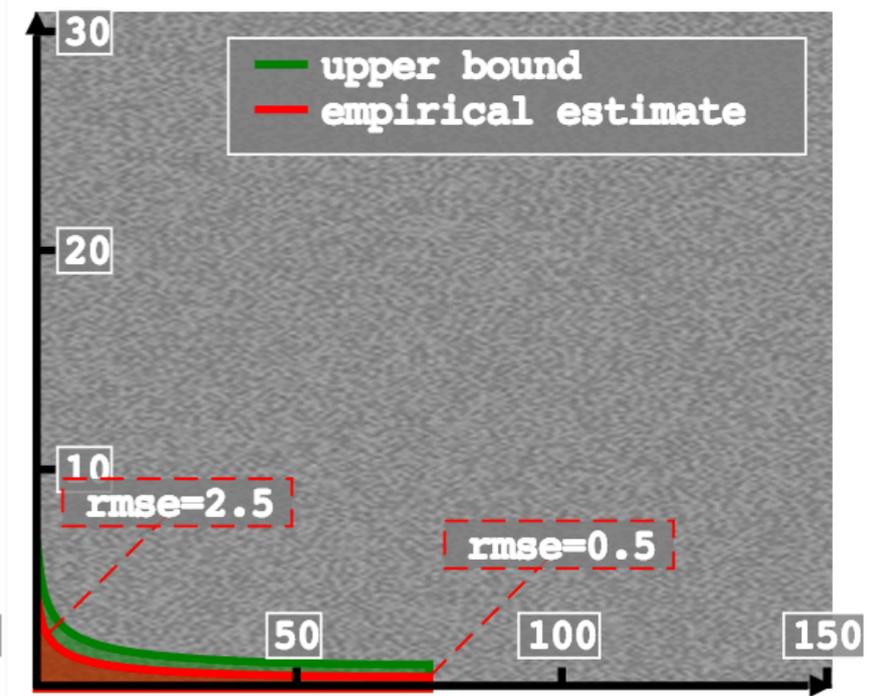
Bound tightness



Sinc



Sinc w/o μ



DFT

➔ Simulated white-noise

$$\rightarrow \sqrt{\mathbb{E}[\text{quant}^2]} = 0.3$$

Bound tightness



Sinc

➔ Textured image



Sinc w/o μ

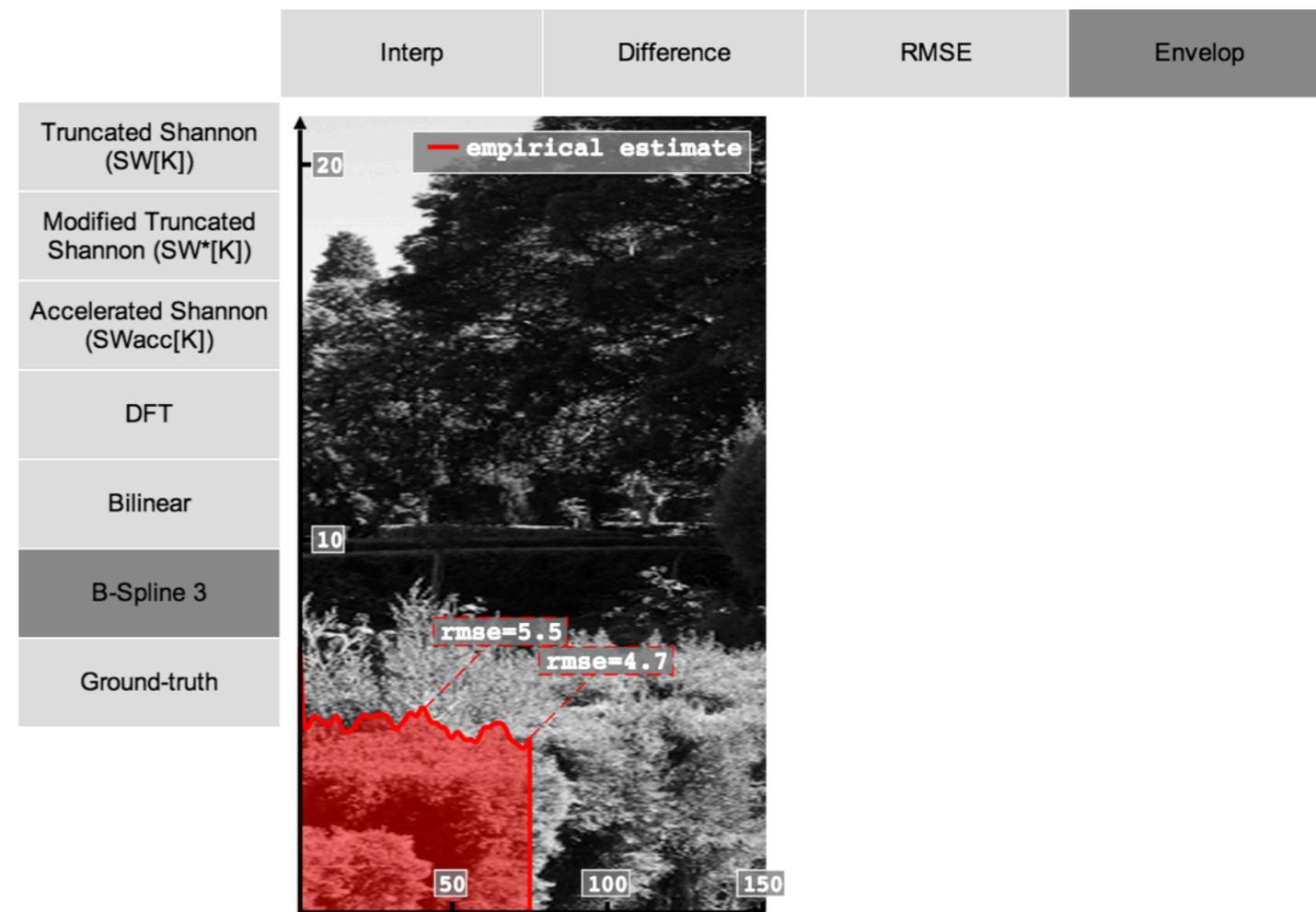


DFT

➔ $\sqrt{\mathbb{E}[\text{quant}^2]} = 0.3$

Other kernels?

- Bound validity
- Bound tightness
- Order of magnitude



➔ online demo:



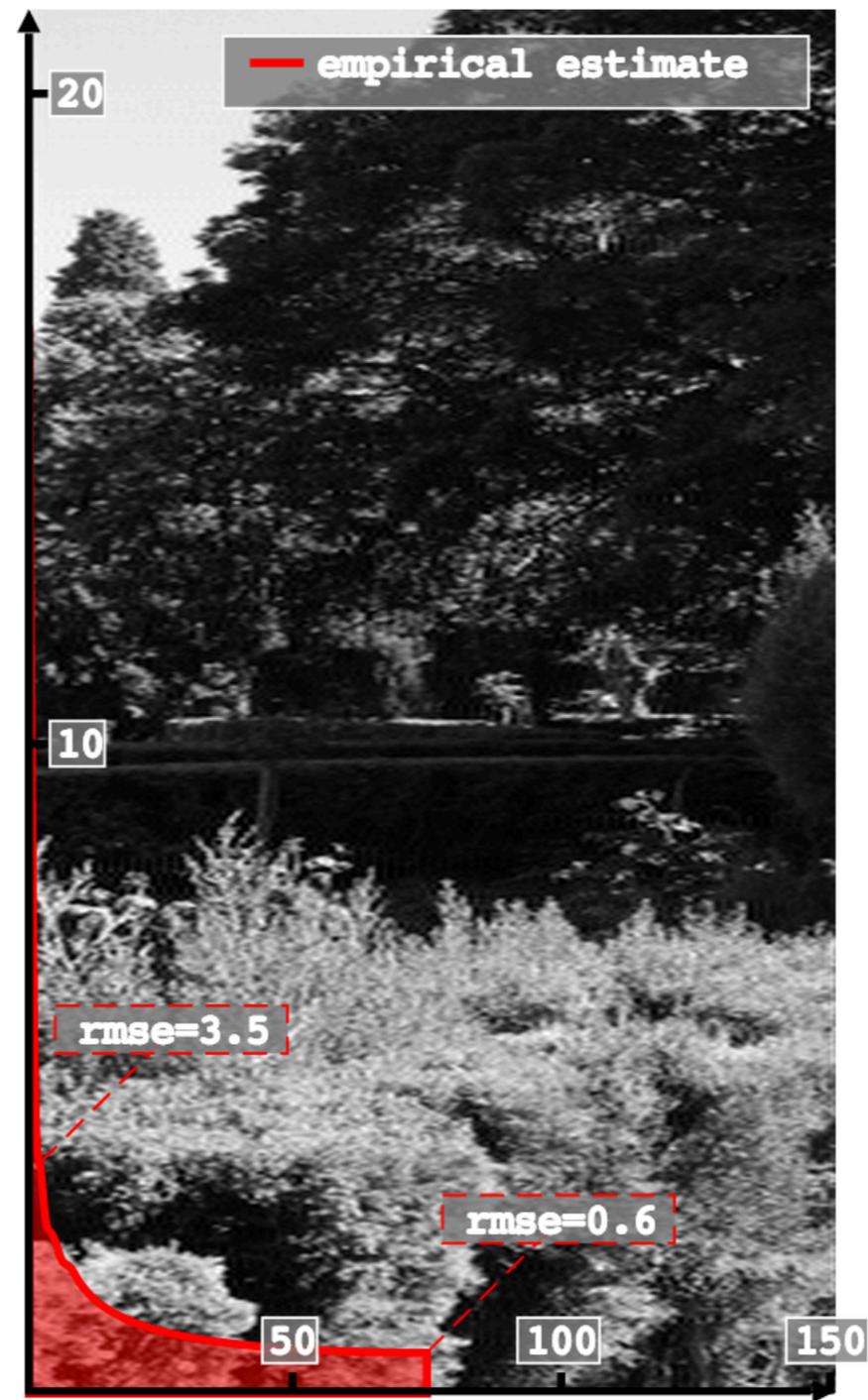
IPOP · Image Processing On Line

Other kernels?



Sinc w/o μ

Other kernels?



Sinc + accel

Other kernels?



Bilinear

Other kernels?



Bicubic

Other kernels?



B-Spline 3

Other kernels?

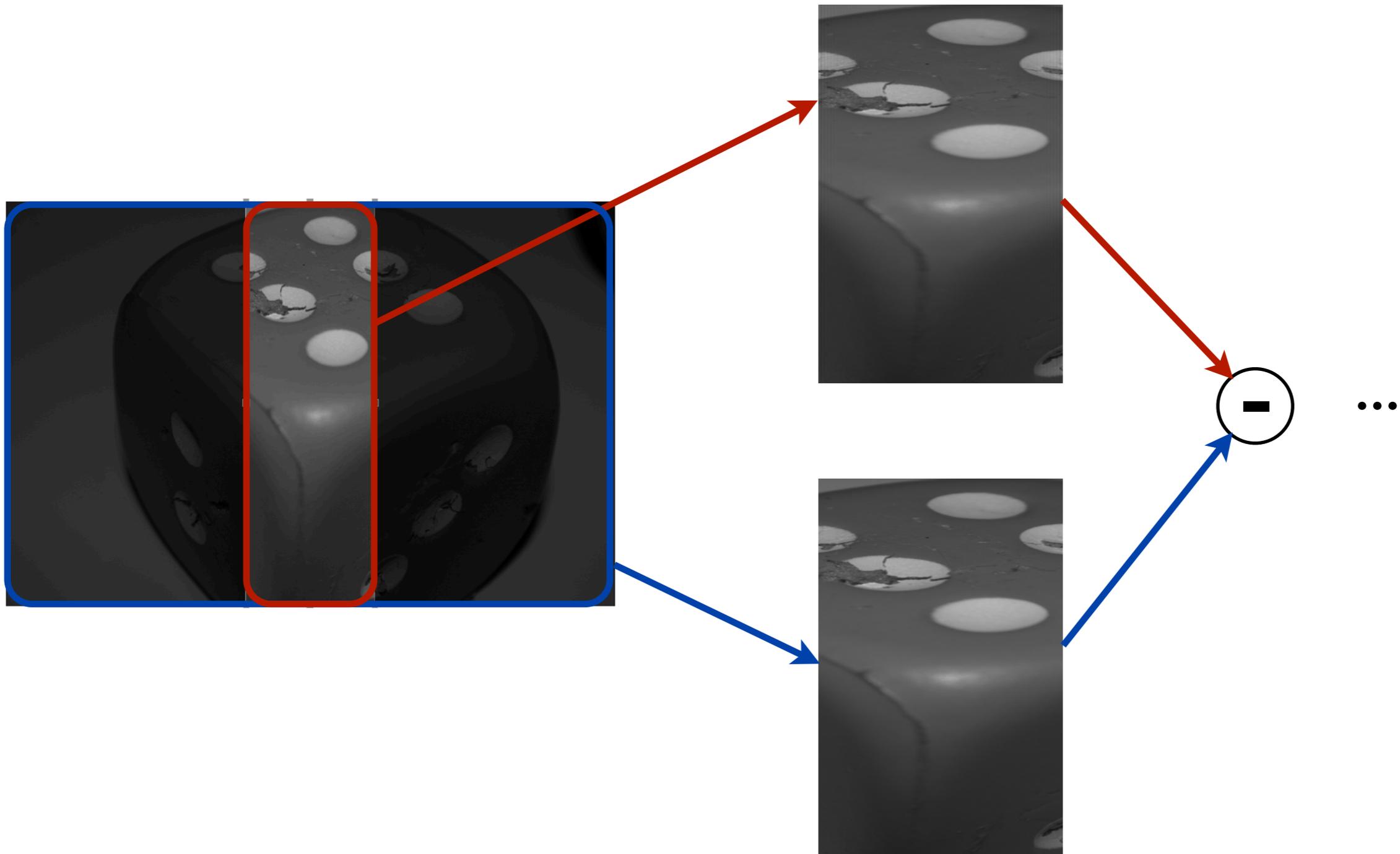


B-Spline I I

Conclusion

- Textures are nasty
- Aliasing is not the worst thing in life
- Is there a hope for image interpolation?

Empirical estimate



Textures

- What's special about images?

