*Solution 1.4: we have by definition*

$$\mathbb{E}(T_n-\mu)^2=\mathbb{E}\left[(\overline{X}_n-\mu)^2 1_{|\overline{X}_n|>n^{-1/4}}\right]+\mu^2\mathbb{P}\left(|\overline{X}_n|\le n^{-1/4}\right).$$

*At $\mu=0$, it simplifies to*

$$\mathbb{E}T_n^2=\mathbb{E}\left[\overline{X}_n^2 1_{|\overline{X}_n|>n^{-1/4}}\right].$$

*The boundedness of $\overline{X}_n$ and Cauchy-Schwarz's inequality lead to*

$$\mathbb{E}T_n^2=\mathbb{E}\left[\overline{X}_n^2 1_{|\overline{X}_n|>n^{-1/4}}\right]\le\sqrt{\mathbb{E}\left[\overline{X}_n^4\right]\mathbb{P}\left(|\overline{X}_n|>n^{-1/4}\right)}\le 2b^2\exp\left(-\frac{\sqrt{n}}{4b^2}\right). \qquad \square$$

5. Show that for any fixed $\mu\neq 0$, $R(T_n,\mu)$ converges to zero at rate $O(1/n)$.

*Solution: thanks to the triangle inequality $|\overline{X}_n-\mu|\ge|\mu|-|\overline{X}_n|$, we deduce*

$$\mathbb{P}\left(|\overline{X}_n|\le|\mu|-n^{-1/4}\right)\le\mathbb{P}\left(|\overline{X}_n-\mu|\ge n^{-1/4}\right)\le 2\exp\left(-\frac{\sqrt{n}}{2b^2}\right).$$

*When $\mu\neq 0$, as soon as $|\mu|>2n^{-1/4}$, we obtain*

$$\mathbb{P}\left(|\overline{X}_n|\le n^{-1/4}\right)\le\mathbb{P}\left(|\overline{X}_n|\le|\mu|-n^{-1/4}\right)\le 2\exp\left(-\frac{\sqrt{n}}{2b^2}\right).$$

*Again, writing out the expression for the quadratic risk*

$$\begin{aligned}\mathbb{E}(T_n-\mu)^2&=\mathbb{E}\left[(\overline{X}_n-\mu)^2 1_{|\overline{X}_n|>n^{-1/4}}\right]+\mu^2\mathbb{P}\left(|\overline{X}_n|\le n^{-1/4}\right)\\&=\mathbb{E}\left[(\overline{X}_n-\mu)^2\right]-\mathbb{E}\left[(\overline{X}_n-\mu)^2 1_{|\overline{X}_n|\le n^{-1/4}}\right]+\mu^2\mathbb{P}\left(|\overline{X}_n|\le n^{-1/4}\right),\end{aligned}$$

*we find that the second and third term decrease much faster than the first one. Hence the conclusion.* $\square$

*Solution 1.5: we lower bound the MSE*

$$\mathbb{E}(T_n-\mu_n)^2=\mathbb{E}\left[(\overline{X}_n-\mu_n)^2 1_{|\overline{X}_n|>n^{-1/4}}\right]+\mu_n^2\mathbb{P}\left(|\overline{X}_n|\le n^{-1/4}\right)\ge\mu_n^2\mathbb{P}\left(|\overline{X}_n|\le n^{-1/4}\right)$$

*as the first term decreases at least as fast as $O(1/n)$. Since we have by the concentration inequality*

$$\mathbb{P}\left(|\overline{X}_n|\le n^{-1/4}\right)\ge\mathbb{P}\left(|\overline{X}_n-\mu_n|\le\frac{1}{2}n^{-1/4}\right)\ge 1-2\exp\left(-\frac{\sqrt{n}}{8b^2}\right),$$

*it follows immediately that*

$$\lim_{n\to\infty}\mu_n^2\mathbb{P}\left(|\overline{X}_n|\le n^{-1/4}\right)=\frac{1}{4\sqrt{n}}$$

*hence the result.* $\square$

*Solution 2.2: Let $\eta$ denote the LRT's threshold and we want it to satisfy*

$$\mathbb{P}\left((\boldsymbol{b}-\boldsymbol{a})^T\Sigma^{-1}Y>\eta|X=0\right)=\alpha.$$

*Note that under $X = 0$, the test statistic is*

$$(\boldsymbol{b}-\boldsymbol{a})^T\Sigma^{-1}Y = (\boldsymbol{b}-\boldsymbol{a})^T\Sigma^{-1}(\boldsymbol{a}+N) \sim \mathcal{N}\left((\boldsymbol{b}-\boldsymbol{a})^T\Sigma^{-1}\boldsymbol{a}, (\boldsymbol{b}-\boldsymbol{a})^T\Sigma^{-1}(\boldsymbol{b}-\boldsymbol{a})\right).$$

*Therefore, let $\gamma := \sqrt{(\boldsymbol{b}-\boldsymbol{a})^T\Sigma^{-1}(\boldsymbol{b}-\boldsymbol{a})}$. The rejection region is*

$$\left\{\boldsymbol{y} \in \mathbb{R}^d,\ \ (\boldsymbol{b}-\boldsymbol{a})^T\Sigma^{-1}(\boldsymbol{y}-\boldsymbol{a}) \geq \gamma F^{-1}(1-\alpha)\right\}$$

*where $F^{-1}(\cdot)$ denotes the quantile of the standard normal distribution.* □

*Solution 2.3: again, let us define the rejection region*

$$R = \{\boldsymbol{y} \in \mathbb{R}^d, \quad (\boldsymbol{b}-\boldsymbol{a})^T\Sigma^{-1}Y > \eta\}.$$

*The type I error associated with the threshold $\eta$ is*

$$\mathbb{P}\left((\boldsymbol{b}-\boldsymbol{a})^T\Sigma^{-1}(Y-\boldsymbol{a}) > \eta - (\boldsymbol{b}-\boldsymbol{a})^T\Sigma^{-1}\boldsymbol{a}|X=0\right)$$

*and its type II error is*

$$\mathbb{P}\left((\boldsymbol{b}-\boldsymbol{a})^T\Sigma^{-1}(Y-\boldsymbol{b}) \leq \eta - (\boldsymbol{b}-\boldsymbol{a})^T\Sigma^{-1}\boldsymbol{b}|X=1\right)$$

*It follows from the requirement that the two errors are equal*

$$\eta - (\boldsymbol{b}-\boldsymbol{a})^T\Sigma^{-1}\boldsymbol{b} = -\eta + (\boldsymbol{b}-\boldsymbol{a})^T\Sigma^{-1}\boldsymbol{a}$$

*that is*

$$\eta = \frac{1}{2}(\boldsymbol{b}-\boldsymbol{a})^T\Sigma^{-1}(\boldsymbol{b}+\boldsymbol{a}). \quad \square$$

*Solution 2.4: it suffices to look at the type I error, which can be written as*

$$\mathbb{P}\left((\boldsymbol{b}-\boldsymbol{a})^T\Sigma^{-1}N > \frac{1}{2}(\boldsymbol{b}-\boldsymbol{a})^T\Sigma^{-1}(\boldsymbol{b}-\boldsymbol{a})\right) = 1 - F\left(\frac{\gamma}{2}\right).$$

*Hence, the stronger the SNR, the smaller the error.* □